PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2012-04

Smoothing and perturbation for some fourth order linear parabolic equations in R^N

Carlos Quesada and Aníbal Rodríguez-Bernal

Mayo-2012

http://www.mat.ucm.es/deptos/ma
e-mail:matemática_aplicada@mat.ucm.es

Smoothing and perturbation for some fourth order linear parabolic equations in $\mathbb{R}^N$ *

Carlos Quesada[†] and Aníbal Rodríguez-Bernal[†] [‡]

[†]Departamento de Matemática Aplicada
Universidad Complutense de Madrid,
Madrid 28040, SPAIN
and
[‡]Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM

Abstract

Using abstract parabolic arguments, we solve the parabolic bi-Laplacian equation in several spaces simultaneously. We can add perturbations to the problem, obtaining a perturbed semigroup, which gives the solution in the scale of spaces, and showing the robustness of the result with respect to the perturbation. For introducing the perturbations, we construct an existence and regularity theory for the unperturbed parabolic bi-Laplacian equation and then add the perturbations. Finally, following the same methods, we consider the problem in bigger space, the uniform Bessel-Lebesgue spaces, and also higher order powers of the Laplacian.

Key words: bi-Laplacian, analytic semigroups, perturbation, semilinear parabolic equations, smoothing, Bessel spaces, uniform spaces.
Mathematical Subject Classification 2010: 35B20, 35B30, 35B35, 35B65, 35G05, 35G16, 35K25, 35K90, 35K91, 47D03, 47D06.

1 Introduction

In this paper we address the solvability of some fourth order linear parabolic equations in $\mathbb{R}^N$. More precisely, we consider

$$\begin{cases} u_t + \Delta^2 u + Pu = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{1.1}$$

*Partially supported by Project MTM2009–07540, MEC and GR58/08 Grupo 920894, UCM, Spain.

with u_0 a suitable initial data defined in $\mathbb{R}^N$ and P a linear perturbation. We will consider space dependent perturbations of the form $Pu := \sum_{a,b} P_{a,b}u$ with

$$P_{a,b}u := D^b(d(x)D^a u) \qquad x \in \mathbb{R}^N \tag{1.2}$$

for some $a, b \in \{0, 1, 2, 3\}$ such that $a+b \leq 3$, where D^a, D^b denote any partial derivatives of order a, b, and a given function $d(x)$ with $x \in \mathbb{R}^N$.

Our main goal is to consider in (1.1) some large classes of initial data u_0 in $\mathbb{R}^N$ as well as to consider wide classes of perturbations. For the latter we will consider classes of coefficients $d(x)$ with weak integrability properties. More precisely, we will assume below that the coefficient $d(x)$ belongs to some locally uniform space $L^p_U(\mathbb{R}^N)$, $1 \leq p \leq \infty$, composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leq C$$

endowed with the norm $\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0 \in \mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}$.

As for the initial data we will consider the standard Lebesgue space, $L^q(\mathbb{R}^N)$, $1 < q < \infty$, or Bessel-Lebesgue spaces $H^{\alpha,q}(\mathbb{R}^N)$, with $1 < q < \infty$, $\alpha \in \mathbb{R}$ and even uniform Bessel spaces $\dot{H}^{\alpha,q}_U(\mathbb{R}^N)$ to be introduced below.

Given such classes of initial data and perturbations we want to find suitable smoothing estimates on the solutions as will be explained below.

Note that for $P = 0$ the solution of problem (1.1) can be described as the convolution of the initial data with the self-similar fundamental kernel for the bi–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [9, 10] and [8, 6].

Recently, results in Bessel-Lebesgue spaces have been proved in [7] for $P \neq 0$. By means of resolvent estimates for $\Delta^2 + P$, the authors proved the well possedness of (1.1) with $Pu = d(x)u$, that is a perturbation with $a, b = 0$. They also found suitable smoothing estimates as the ones we will find below.

Here, instead of relying on elliptic resolvent estimates for operators $\Delta^2 + P$, with P as in (1.2), we rely on a more abstract "parabolic" argument developed in [14] and applied there to parabolic equations with second order elliptic operators. With this approach we consider a simpler problem, the one with $P = 0$, that we can solve in several spaces simultaneously. That is, we consider a semigroup of solutions defined on a scale of spaces. For such simpler problem we start by proving suitable smoothing estimates on the spaces of the scale. Then we consider a suitable perturbation, P, that acts between two spaces on the scale. With these ingredients the abstract results in [14] allow to obtain a perturbed semigroup that corresponds to the equation with $P \neq 0$. Such perturbed semigroup inherits some of the smoothing estimates of the original one in some of the spaces of the scale which are determined by the perturbation P itself.

Another important result that we are able to stablish using the tools developed in [14] is that of the robustness with respect to the perturbation. In this direction we are able to prove two important results. First, we show that all constants involved in the smoothing estimates of the perturbed semigroups, including the exponential bounds on them, are bounded uniformly for bounded families of perturbations (i.e. for families of coefficients $d(x)$ as in (1.2) which are bounded in the uniform space $L^p_U(\mathbb{R}^N)$). Second, we prove that the perturbed semigroups obtained as above, continuously depend on the perturbation. That is, if the coefficients $d(x)$ depend on a parameter and converge in the space $L^p_U(\mathbb{R}^N)$, then the corresponding semigroups converge in norm.

As mentioned above this approach was applied in [14] to second order parabolic equations in bounded and unbounded domains, allowing perturbations in the equation and in the boundary conditions.

In this paper however we carry out these ideas to fourth order parabolic equations in $\mathbb{R}^N$ as in (1.1). Hence we need to develop an existence and regularity theory in suitable scales of spaces for the parabolic bi–Laplacian equation, i.e. (1.1) with $P = 0$, in order to later introduce the perturbations. For this we use as much as informations as we have about the heat equation $u_t - \Delta u = 0$, in $\mathbb{R}^N$ and use that Δ^2 is the square operator of $-\Delta$. In particular we show that the same scales of spaces available for $-\Delta$ can be used for (1.1). In such scales suitable smoothing estimates for (1.1) with $P = 0$ are obtained.

We now state one of the main results that we prove below, see Theorem 5.10. Note that this result applies in the Bessel–Lebesgue scale. A similar one, with technical differences, holds in the uniform Bessel scale, see Theorem 6.8.

Theorem 1.1 *Let $P_{a,b}$ be as in (1.2) with $k, a, b \in \{0, 1, 2, 3\}$, $k = a + b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-k}$.*

Then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q, a, b) \subset (-1+\frac{a}{4}, 1-\frac{b}{4})$ containing $(-1 + \frac{a}{4} + \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem

$$\begin{cases} u_t + \Delta^2 u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

for $1 < q \leq r \leq \infty$, with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, a, b)$, $\gamma \geq \gamma'$ and for any $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Observe that in the theorem above just one perturbation $P_{a,b}$ is considered for the bi–Laplacian operator. Also note the ranges of spaces for which we can solve the equation are determined by the base space in terms of $1 < q < \infty$, the integrability p of the coefficient $d(x)$ and the order of derivatives a, b. Several perturbations can be thus combined together, although not all combinations are allowed. We discuss below a general procedure to determine whether or not two given perturbations can be combined together; see Remark 5.11.

The paper is organized as follows. In Section 2 we recall the main results in [14] that will be used in this paper. Note that Theorem 2.3 is stated containing a case not considered in [14] but that will be required further below.

In Section 3 we collect the construction of suitable scales of spaces for sectorial operators (that is, negative of generators of analytic semigroups) in Banach spaces. For this we follow the general constructions in [1] and construct both an interpolation/extrapolation scale and a fractional power scale. On these scales the operator defines a strongly continuous analytic semigroup with suitable smoothing estimates, see Propositions 3.6 and 3.7.

In Section 4 we assume that a sectorial operator as in Section 3 is such that its square is also sectorial. Then we show that both the interpolation/extrapolation scale of the operator and its square coincide after a suitable labeling. We also obtain the corresponding smoothing estimates for the semigroup of the square of the operator; see Propositions 4.3 and 4.4. In this section the results in [12] will play an essential role.

Then we apply all these abstract results to (1.1). In Section 5 we prove that Δ^2 defines an analytic semigroup in the scales of Lebesgue and Bessel–Lebesgue spaces which satisfy suitable smoothing estimates; see Lemma 5.2. Then using the results in Section 2 we are able to add perturbations to the equation along the lines described above, see Lemma 5.5, Lemma 5.7 and Theorem 5.10. Some extension to fractional–like derivatives in (1.2) can be found in Theorem 5.12. In this case a, b are nonnegative real and $0 \leq a + b < 4$.

The same strategy is carried out in Section 6 for (1.1) in the uniform Bessel–Lebesgue scale. Such scale was used for linear and nonlinear heat equations in [3, 5]. Such spaces are useful because, among other properties, they are very large spaces whose functions do not satisfy any smallness behavior at infinity and contain the standard Bessel–Lebesgue spaces as closed subspaces. After some result on these spaces in Proposition 6.1 that complements the ones in [3], we obtain resolvent estimates for the Laplacian operator that prove that it is sectorial and that allows us to use the results in Section 4 to handle the bi–Laplacian in uniform spaces. Then in Lemma 6.4 we show that the bi–Laplacian parabolic equation defines an analytic semigroup with suitable smoothing estimates in the uniform Bessel–Lebesgue scale. Then in Lemma 6.6 and Theorem 6.8 we introduce the perturbations and prove an analogous result to Theorem 1.1 in this scale. Note that since uniform spaces are not reflexive (even for $q = 2$) we can only consider the case $b = 0$ in (1.2) and in Theorem 1.1, see Theorem 6.8.

Finally, in Section 7 we show how to obtain all the results in Sections 4, 5 and 6 for other powers of the Laplacian $(-\Delta)^m$, $m \in \mathbb{N}$ as the main part in the elliptic operator.

2 Some previous results

We recall some results from [14] that will be needed later on. Let $\{X_\alpha\}_{\alpha\in I}$ be a family of Banach spaces, with α in an interval I, endowed with a norm $\|\cdot\|_\alpha$. Let $S(t)$ be a semigroup on a scale $\{X_\alpha\}_{\alpha\in I}$, such that

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(X_\beta,X_\alpha)} \leq \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \qquad \forall \quad 0 < t \leq 1 \tag{2.1}$$

for all $\alpha, \beta \in I$, $\alpha \geq \beta$ for some constant $M_0(\beta, \alpha) > 0$.

Now, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X_\alpha, X_\beta). \tag{2.2}$$

$$0 \leq \alpha - \beta < 1. \tag{2.3}$$

We will sometimes use "nested" spaces, that is, for all $\alpha, \beta \in I$ with $\alpha \geq \beta$ we have

$$X_\alpha \subset X_\beta \tag{2.4}$$

with continuous inclusion and the norm of the inclusion will be denoted $\|i\|_{\alpha,\beta}$. This will be explicitly stated when used.

Consider the perturbed problem

$$u(t; u_0) = S(t)u_0 + \int_0^t S(t-\tau)Pu(\tau; u_0)\, d\tau, \qquad t > 0, \tag{2.5}$$

which corresponds to solving the problem $u_t + Au = Pu$, where $-A$ is the infinitesimal generator of the semigroup $S(t)$.

The following result is taken from [14, Proposition 10] and states the existence of a perturbed semigroup defined by (2.5).

Theorem 2.1 *Assume (2.1), (2.2), and (2.3). Then for every $R_0 > 0$ and every*

$$P \in \mathcal{L}(X_\alpha, X_\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X_\alpha,X_\beta)} \leq R_0$$

and for every $\gamma, \gamma' \in I$ such that

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma, \tag{2.6}$$

there exist constants $\omega = \omega(\gamma, \gamma', R_0) \geq 0$ and $M_0 = M_0(\gamma, \gamma', R_0)$ such that, for $t > 0$, there exists a unique solution of (2.5), which defines a mapping from X_γ into $X_{\gamma'}$ as

$$S_P(t)u_0 := u(t; u_0), \quad \textit{for all} \quad t > 0$$

such that

$$\|S_P(t)u_0\|_{\gamma'} \leq M_0 \mathrm{e}^{\omega t} t^{-(\gamma' - \gamma)} \|u_0\|_\gamma, \quad \gamma' \geq \gamma. \tag{2.7}$$

In particular for any $\gamma \in [\beta, \alpha]$, $S_P(t) \in \mathcal{L}(X_\gamma)$ and it is a semigroup of linear continuous operators in X_γ.

The same is true for any $\gamma \in E(\alpha)$, if the scale is nested.

Now we turn into the continuity of the perturbed semigroup with respect to the perturbation. With the setting above, assume that we have two perturbations

$$P_i \in \mathcal{L}(X_\alpha, X_\beta), \quad i = 1, 2, \quad 0 \leq \alpha - \beta < 1.$$

Our goal is then to compare semigroups $S_{P_i}(t)$, $i = 1, 2$. Hence assume

$$\|P_i\|_{\mathcal{L}(X_\alpha, X_\beta)} \leq R_0 \quad i = 1, 2$$

for some $R_0 > 0$. Also, consider the existence and regularity intervals as in (2.6)

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Consider then an initial data $u_0 \in X_\gamma$, and the corresponding solutions of the perturbed problem

$$u^i(t; u_0) = S_{P_i}(t)u_0 = S(t)u_0 + \int_0^t S(t - \tau) P_i u^i(\tau; u_0)\, d\tau, \qquad t > 0.$$

Then we have the following continuity result, see [14, Theorem 14].

Theorem 2.2 *With the notations above, for any $R_0 > 0$, there exists a sufficiently small T_0 such that for all perturbations P_i, $i = 1, 2$, such that $\|P_i\|_{\mathcal{L}(X_\alpha, X_\beta)} \leq R_0$,*

$$\|S_{P_1}(t) - S_{P_2}(t)\|_{\mathcal{L}(X_\gamma, X_{\gamma'})} \leq \frac{L(T_0, R_0)}{t^{\gamma' - \gamma}} \|P_1 - P_2\|_{\mathcal{L}(X_\alpha, X_\beta)}, \qquad \textit{for all} \quad 0 < t \leq T_0$$

and for every $T > T_0$

$$\|S_{P_1}(t) - S_{P_2}(t)\|_{\mathcal{L}(X_\gamma, X_{\gamma'})} \leq L(T, T_0, R_0) \|P_1 - P_2\|_{\mathcal{L}(X_\alpha, X_\beta)}, \qquad \textit{for all} \quad T_0 < t \leq T.$$

Finally we will also need the following result about the analyticity of the semigroup defined by (2.5). Note that the first part of the Theorem below is taken from [14, Theorem 12], but the second part we introduce here will be also needed further below.

Theorem 2.3 *Assume the scale is nested, that is, (2.4), and that for any $\gamma \in I$, if $-A$ denotes the infinitesimal generator of $S(t)$ in X_γ, then its domain is given by $D(A) = X_{\gamma+1}$.*

Also assume the scale satisfies either one of the following interpolation properties:
i) If Y is a Banach space and $T \in \mathcal{L}(X_\gamma, Y)$ and $T \in \mathcal{L}(X_{\gamma'}, Y)$ then $T \in \mathcal{L}(X_{\theta\gamma+(1-\theta)\gamma'}, Y)$ for $\theta \in [0,1]$ and

$$\|T\|_{\mathcal{L}(X_{\theta\gamma+(1-\theta)\gamma'},Y)} \leq \|T\|^{\theta}_{\mathcal{L}(X_\gamma,Y)} \|T\|^{1-\theta}_{\mathcal{L}(X_{\gamma'},Y)}. \tag{2.8}$$

ii) The following condition is satisfied for any $\gamma, \gamma' \in I$ and $0 < \theta < 1$

$$\|u\|_{X_{\theta\gamma+(1-\theta)\gamma}} \leq C \|u\|^{\theta}_{X_\gamma} \|u\|^{1-\theta}_{X_{\gamma'}}. \tag{2.9}$$

Finally, as in Theorem 2.1, assume that for some fixed $\alpha \geq \beta$, with $0 \leq \alpha - \beta < 1$ we have a linear perturbation satisfying

$$P \in \mathcal{L}(X_\alpha, X_\beta) \quad \textit{with} \quad \|P\|_{\mathcal{L}(X_\alpha,X_\beta)} \leq R_0.$$

Then, there exists some $0 < \omega_0 = \omega_0(R_0)$ such that for any $Re(\lambda) \geq \omega_0$ and any $\gamma \in (\alpha-1, \beta)$ the operator $A + \lambda I - P$, between $X_{\gamma+1}$ and X_γ, is invertible and

$$\|\big(A + \lambda I - P\big)^{-1}\|_{\mathcal{L}(X_\gamma,X_\gamma)} \leq \frac{C}{|\lambda|}, \qquad Re(\lambda) \geq \omega_0$$

and

$$\|\big(A + \lambda I - P\big)^{-1}\|_{\mathcal{L}(X_\gamma,X_{\gamma+1})} \leq C, \qquad Re(\lambda) \geq \omega_0$$

where C is independent of P and λ.

In particular, for every $\gamma \in (\alpha-1, \beta)$, the semigroup $S_P(t)$ in X_γ in Theorem 2.1 is analytic.

Proof. The proof of part i) can be found in [14, Theorem 12].

Under the assumption in ii) the same proof remains unchanged up to the point where for all $\gamma \in I$ the following inequalities are obtained

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X_\gamma,X_\gamma)} \leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \geq \omega$$

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X_{\gamma+1},X_{\gamma+1})} \leq \frac{C}{|\lambda|}, \qquad \mathrm{Re}(\lambda) \geq \omega \tag{2.10}$$

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X_\gamma,X_{\gamma+1})} \leq C, \qquad \mathrm{Re}(\lambda) \geq \omega.$$

At this point we proceed as follows. For any $\gamma \in I$ and $\tilde{\gamma} \in (\gamma, \gamma+1)$ we have that $\gamma + 1 \in (\tilde{\gamma}, \tilde{\gamma}+1)$ and thus, using (2.10) and (2.9), we get for $\mathrm{Re}(\lambda) \geq \omega$

$$\|(A+\lambda)^{-1}u\|_{\gamma+1} \leq \|(A+\lambda)^{-1}u\|_{\tilde{\gamma}}^{\theta}\|(A+\lambda)^{-1}u\|_{\tilde{\gamma}+1}^{1-\theta} \leq \frac{C}{|\lambda|^{\theta}}\|u\|_{\tilde{\gamma}}^{\theta}\|u\|_{\tilde{\gamma}}^{1-\theta} = \frac{C}{|\lambda|^{\theta}}\|u\|_{\tilde{\gamma}}$$

for θ such that $\gamma + 1 = \theta\tilde{\gamma} + (1-\theta)(\tilde{\gamma}+1)$, that is, $\theta = \tilde{\gamma} - \gamma$. Hence we get

$$\|(A+\lambda)^{-1}\|_{\mathcal{L}(X_{\tilde{\gamma}}, X_{\gamma+1})} \leq \frac{C}{|\lambda|^{\tilde{\gamma}-\gamma}}, \qquad \mathrm{Re}(\lambda) \geq \omega.$$

Now the proof concludes as in [14, Theorem 12]. ■

3 Scales of spaces for sectorial operators

In this section, we construct suitable scales of spaces for sectorial operators in Banach spaces. These constructions follow [1] and, in view of the applications later in this paper, we particularize for the scales of complex interpolation–extrapolation spaces and the scale of fractional power spaces.

Following [1], let E_0, E_1 be Banach spaces with continuous inclusion $E_1 \subset E_0$ and consider the class $\mathcal{H}(E_1, E_0)$ of linear operators in E_0, with domain E_1 such that if $A_0 \in \mathcal{H}(E_1, E_0)$, then $-A_0$ generates a strongly continuous analytic semigroup in E_0, $\{e^{-A_0 t}; t \geq 0\}$.

For generators of analytic semigroups we have the following well known definitions.

Definition 3.1
i) [11, Definition 1.3.1 pg 18]. A closed operator in a Banach space E_0, A_0, with domain $D(A_0)$, is sectorial if there exists a sector

$$S_{a,\phi} = \{z \in \mathbb{C} : \phi \leq |arg(z-a)| \leq \pi,\ z \neq a\} \subset \rho(A_0) \tag{3.1}$$

for some $a \in \mathbb{R}$ and $\phi \in (0, \pi/2)$, such that

$$\|(A_0 - \lambda)^{-1}\|_{E_0} \leq M|\lambda - a|^{-1} \qquad \text{for all } \lambda \in S_{a,\phi}. \tag{3.2}$$

ii)[1, Section 1.2]. $\mathcal{H}(E_1, E_0) = \bigcup_{\substack{\kappa \geq 1 \\ \omega > 0}} \mathcal{H}(E_1, E_0, \kappa, \omega)$, where $A_0 \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ if $-\omega + A_0 \in \mathcal{L}is(E_1, E_0)$ and

$$\kappa^{-1} \leq \frac{\|(A_0 - \lambda)x\|_{E_0}}{|\lambda|\|x\|_{E_0} + \|x\|_{E_1}} \leq \kappa, \qquad Re(\lambda) \leq -\omega \qquad x \in E_1. \tag{3.3}$$

The following result establishes the equivalence between both definitions.

Proposition 3.2 *Both definitions i) and ii) in Definition 3.1 are equivalent.*

Proof. i)⇒ii)

Define $E_1 := D(A_0)$ with the graph norm, that is

$$\|\cdot\|_{E_1} := \|\cdot\|_{G(A_0)} := \|\cdot\|_{E_0} + \|A_0(\cdot)\|_{E_0}.$$

Note that [1, Remark 1.2.1 pg 11] proves (3.3) provided we prove

$$|\lambda|\|x\|_{E_0} \leq \kappa\|(A_0 - \lambda)x\|_{E_0} \qquad \mathrm{Re}(\lambda) \leq -\omega \qquad x \in E_1.$$

Thus from (3.2) we get

$$M|\lambda - a|\|x\|_{E_0} \leq \|(A_0 - \lambda)x\|_{E_0} \qquad \text{for all } \lambda \in S_{a,\phi},\ x \in D(A_0) = E_1.$$

Now, if we take $\omega > 0$ such that $-\omega <\mathrm{Re}(a)$, then $-\omega \in \rho(A_0)$, thus $-\omega + A_0 \in \mathcal{L}is(E_1, E_0)$ and $\frac{|\lambda|}{|\lambda - a|} \leq \tilde{M}$ for all $\mathrm{Re}(\lambda) \leq -\omega$. Hence

$$\tilde{M}M|\lambda|\|x\|_{E_0} \leq \|(A_0 - \lambda)x\|_{E_0} \qquad \mathrm{Re}(\lambda) \leq -\omega \qquad x \in E_1.$$

ii)⇒i)

For proving this, we first use Proposition [1, I.1.4.1, pg 15], which read in terms of our notation, states that if $A_0 \in \mathcal{H}(E_1, E_0, \kappa, \omega)$ then there exist $\kappa \geq 1$, $\omega > 0$, $-\omega_0 \in (-\omega, 0)$ and $\theta \in (0, \pi/2)$ such that we have that

$$\frac{1}{5\kappa} \leq \frac{\|(A_0 - \lambda)x\|_{E_0}}{|\lambda|\|x\|_{E_0} + \|x\|_{E_1}} \leq 5\kappa \qquad x \in E_1$$

for $\lambda \in \Sigma_{-\omega_0,\theta} := \{|\arg(z - \omega_0) \leq \theta + \pi/2|\} \subset \rho(A_0)$.

Note that taking $a = -\omega_0$ and $\phi = \frac{\pi}{2} - \theta$ we define $S_{a,\phi} = \Sigma_{-\omega_0,\theta}$ and we just need to check that

$$\|(A_0 - \lambda)^{-1}\|_{E_0} \leq M|\lambda - a|^{-1} \qquad \lambda \in S_{a,\phi}$$

From $\frac{1}{5\kappa} \leq \frac{\|(A_0-\lambda)x\|_{E_0}}{|\lambda|\|x\|_{E_0}+\|x\|_{E_1}}$ we get for $\lambda \in S_{a,\phi}$

$$C|\lambda|\|x\|_0 \leq \|(A_0 - \lambda)x\|_0 \qquad x \in E_1$$

which, taking $y = (A_0 - \lambda)x$, reads

$$\|(A_0 - \lambda)^{-1}y\|_{E_0} \leq \frac{C}{|\lambda|}\|y\|_{E_0}$$

and since $\frac{|\lambda+\omega_0|}{|\lambda|} \leq \tilde{C}$ for all $\lambda \in S_{a,\phi}$, we get

$$\|(A_0 - \lambda)^{-1}y\|_{E_0} \leq \frac{C\tilde{C}}{|\lambda + \omega_0|}\|y\|_{E_0} := \frac{M}{|\lambda - a|}\|y\|_{E_0}.$$

■

Note that for $A_0 \in \mathcal{H}(E_1, E_0)$, we define

$$\text{type}(A_0) = -\inf\{\text{Re}(\sigma(A_0))\}$$

and observe that this quantity will play an important role in the estimates for semigroups below, see e.g. (3.12). For details on this definition see [1, pg. 17, pg. 34 and II.5.1.2, pg. 69], noting that there, the notation is slightly different.

In what follows we will momentarily assume that

$$0 \in \rho(A_0). \tag{3.4}$$

With this it can be proved that the norm $\|\cdot\|_{E_1}$ is equivalent $\|A_0 \cdot\|_{E_0}$, and we can start a recurring construction as follows.

Consider $E_2 := D(A_1) = \{u \in E_1,\ A_1 u \in E_1\}$ where $A_1 : E_2 \hookrightarrow E_1$ is the realization (and also the closure) of A_0 in E_1 and endowed with the norm $\|\cdot\|_{E_2} = \|A_1 \cdot\|_{E_1}$.

We can iterate this process to get a discrete scale of Banach spaces $\{E_n, n \in \mathbb{N}\}$ and the realizations of A_0 in E_n, which we denote by A_n, satisfy $A_n \in \mathcal{H}(E_{n+1}, E_n)$ and are isometric isomorphisms from $E_{n+1} \to E_n$, see [1, V.1.2.1, pg. 256].

For the construction of the negative side of the scale, define E_{-1} as the completion of E_0 relatively to the norm $\|\cdot\|_{E_{-1}} := \|A_0^{-1} \cdot\|_{E_0}$, which is a Banach space such that $E_0 \hookrightarrow E_{-1}$ densely and A_{-1} is the unique continuous extension of A_0, which is an isometric isomorphism from $E_0 \to E_{-1}$. This extension is called again the realization of A_0 in E_{-1}.

Again, we iterate the process of completion with the norm generated by the new operator and we get a negative discrete scale $\{E_{-n}, n \in \mathbb{N}\}$ and $A_{-n} \in \mathcal{H}(E_{-n+1}, E_{-n})$, where A_{-n} denotes the realization of A_0, the closure of A_{-n+1} in E_{-n} and is an isometric isomorphism from $E_{-n+1} \to E_{-n}$ see [1, V.1.3.2, pg. 263] and the comments on [1, pg. 264].

So we have a two-sided discrete nested scale ([1, V.1.3.4, pg 264]):

$$\{E_k,\ k \in \mathbb{Z}\}, \qquad A_k \in \mathcal{H}(E_{k+1}, E_k) \tag{3.5}$$

where A_k denotes the realization of A_0, the closure of A_{k+1} in E_k and is an isometric isomorphism from $E_{k+1} \to E_k$ which satisfies

$$\rho(A_k) = \rho(A_0) \quad k \in \mathbb{Z}. \tag{3.6}$$

In order to have a better description of the negative scale we can use dual spaces as follows, provided E_0 is reflexive.

Assume E_0 is reflexive and let $E_0^\sharp := E_0'$ the dual space and $E_1^\sharp := D(A_0^\sharp)$, where $A_0^\sharp : E_1^\sharp \subset E_0^\sharp \hookrightarrow E_0^\sharp$ is the adjoint operator of A_0, which satisfies $A_0^\sharp \in \mathcal{H}(E_1^\sharp, E_0^\sharp)$, see [1, I.1.2.3, pg. 13].

Then, we repeat the process above and construct a discrete scale $\{E_n^\sharp;\ n \in \mathbb{N}\}$, which can be identified with the original one by

$$E_{-n} = (E_n^\sharp)' \quad \textit{and} \quad A_{-n} = (A_n^\sharp)' \qquad n \in \mathbb{N}, \tag{3.7}$$

where the dashes denote the duals, see [1, V.1.4.9, pg. 272].

Now we construct intermediate spaces between the discrete scale $\{E_k,\ k \in \mathbb{Z}\}$ following two different procedures.

3.1 Construction of the interpolation-extrapolation scale for A_0

Recall that if a Banach space, say G, is densely included in other Banach space, H, they are said to be an *interpolation couple.* Also, an *interpolation functor* of exponent $0 < \theta < 1$, $[\cdot,\cdot]_\theta$, is a map such that for two given interpolation couples G_0, G_1 and H_0, H_1, we have Banach spaces $G_\theta = [G_1, G_0]_\theta$ and $H_\theta = [H_1, H_0]_\theta$ such that $G_1 \subset G_\theta \subset G_0$, $H_1 \subset H_\theta \subset H_0$ and for $A \in \mathcal{L}(G_0, H_0) \cap \mathcal{L}(G_1, H_1)$, then $A \in \mathcal{L}(G_\theta, H_\theta)$ and

$$\|A\|_{\mathcal{L}(G_\theta,H_\theta)} \leq \|A\|^{1-\theta}_{\mathcal{L}(G_0,H_0)} \|A\|^{\theta}_{\mathcal{L}(G_1,H_1)}, \tag{3.8}$$

see [16].

Remark 3.3 *There are many interpolation functors that can be used here, but in particular we choose complex interpolation for simplicity and because in the applications to (1.1) it leads to a very convenient scale of spaces.*

Starting with the discrete scale (3.5) and taking the complex interpolation method, we proceed as in [1, V.1.5.1, pg. 275] to obtain the spaces

$$E_\alpha := E_{k+\theta} := [E_{k+1}, E_k]_\theta, \quad \theta \in (0,1) \quad k \in \mathbb{Z}, \tag{3.9}$$

and the operator A_α as the interpolation of A_{k+1} and A_k, as in (3.8). Thus we obtain the continuous nested interpolation scale

$$\{E_\alpha,\ \alpha \in \mathbb{R}\}, \qquad A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha) \tag{3.10}$$

and A_α is an isometry from $E_{\alpha+1}$ into E_α. Note that if $\alpha > \beta$, E_α is densely included in E_β and A_α is the realization of A_0 in E_α. Moreover, for every $\alpha \in \mathbb{R}$

$$\rho(A_\alpha) = \rho(A_0), \tag{3.11}$$

see [1, V.1.1.2.e), pg. 252].

Now, since $A_\beta \in \mathcal{H}(E_{\beta+1}, E_\beta)$, $-A_\beta$ generates an analytic semigroup in E_β with the property [1, V.2.1.3, pg. 289]:

$$\|e^{-A_\beta t}\|_{\mathcal{L}(E_\beta,E_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R}, \alpha \geq \beta \tag{3.12}$$

for any $\sigma > \text{type}(A_0)$ and $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, we can interpolate in the dual scale $\{E^\sharp_n, n \in \mathbb{Z}\}$ as well. We take again the complex interpolation $[\cdot,\cdot]_\theta$, and the negative intermediate spaces can be identified with the dual of the positive ones as

$$E_{-\alpha} = (E^\sharp_\alpha)' \quad \textit{and} \quad A_{-\alpha} = (A^\sharp_\alpha)' \quad \text{for } \alpha > 0, \tag{3.13}$$

see [1, V.1.5.12, pg. 282]. Also, the semigroup in the spaces of the negative side can be identified with the duals by [1, V.2.3.2, pg. 298]:

$$e^{-A_{-\alpha} t} = (e^{-A^\sharp_\alpha t})' \quad \alpha > 0. \tag{3.14}$$

Note that the semigroups in (3.12) are extensions or restrictions of each other one, that is, given $\alpha \geq \beta$, then

$$e^{-A_\beta t}|_{E_\alpha} = e^{-A_\alpha t}, \quad t \geq 0.$$

For details see Lemma [1, V.2.1.2]. Hence, we have the following.

Definition 3.4 *Under the assumptions above we say that the operator A_0 defines an analytic semigroup $S_{A_0}(t)$ in the interpolation scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ in the sense that*

$$S_{A_0}(t)|_{E_\alpha} = e^{-A_\alpha t}, \qquad \forall \alpha \in \mathbb{R}.$$

Observe that

$$\|S_{A_0}(t)\|_{\mathcal{L}(E_\beta, E_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R}, \alpha \geq \beta$$

for any $\sigma > \text{type}(A_0)$ and $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

Remark 3.5 *Note that we could have taken any other interpolation functor as long as it has the reiteration property (as the complex interpolation does)*

$$[E_\alpha, E_\beta]_\eta = E_{(1-\eta)\alpha+\eta\beta} \quad 0 < \eta < 1, \quad \alpha, \beta \in \mathbb{R}$$

such as real interpolation, and the scale would have had the same properties (3.9), (3.10), (3.11) and (3.12). But then we would have had to use the associated dual interpolation functor of it for the negative part of the scale, to obtain (3.13) and (3.14). For more information see [1, V.1.5.11 pg. 282].

Now we construct the interpolation scale and the semigroup in the scale, as in Definition 3.4, without assuming (3.4).

Proposition 3.6 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and take c such that $0 \in \rho(A_0 + cI)$.*

Then the scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ generated by $A_0 + cI$, as above, is independent of c and for any $\alpha \in \mathbb{R}$, the realization of A_0 in E_α, denoted as A_α, satisfies

$$A_\alpha \in \mathcal{H}(E_{\alpha+1}, E_\alpha)$$

and for all $\alpha \in \mathbb{R}$

$$\rho(A_\alpha) = \rho(A_0).$$

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ such that $S_{A_0}(t)|_{E_\alpha} = e^{-A_\alpha t}$, $\alpha \in \mathbb{R}$, and satisfies

$$\|S_{A_0}(t)\|_{\mathcal{L}(E_\beta, E_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t > 0,\ \alpha \geq \beta \in \mathbb{R}$$

for any $\sigma > type(A_0)$.

Furthermore if E_0 is reflexive, then $E_{-\alpha} = (E_\alpha^\sharp)'$, $A_{-\alpha} = (A_\alpha^\sharp)'$ for $\alpha > 0$, and

$$e^{-A_{-\alpha} t} = (e^{-A_\alpha^\sharp t})'.$$

Proof. If $0 \in \rho(A_0)$, the construction has been carried above.

If $0 \notin \rho(A_0)$, there exists $c \in \mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $0 \in \rho(\tilde{A}_0)$, so we can perform the construction above for the operator $\tilde{A}_0$. Note that the corresponding scale of spaces is independent of c because the interpolation scale is only determined by the spaces $\{E_k\}_{k\in\mathbb{Z}}$, and these spaces have equivalent norms independently of the c chosen.

Thus, with $\tilde{A}_\alpha = A_\alpha + cI$ in E_α and applying standard arguments in [13] or [11] we obtain that

$$e^{-A_\alpha t} = e^{-ct} e^{-\tilde{A}_\alpha t}$$

and the result follows. ■

3.2 Construction of the fractional power scale for A_0

Now, starting again with the discrete scale (3.5), we construct a fractional power scale $\{F_\alpha\}_{\alpha\in\mathbb{R}}$ following [1]. See also [11] and [12]. For this we will also assume for a moment that

$$(-\infty, 0] \subset \rho(A_0). \tag{3.15}$$

Since the intermediate spaces between the integer scale (3.5) might be different to the ones in the previous section, see Remark 3.8 below, we denote now

$$F_k = E_k \quad \text{for } k \in \mathbb{Z}.$$

We first construct the positive fractional power scale. Using (3.15), the resolvent estimate in the sector (see Proposition 3.1) and integrating on a curve which surrounds the sector (3.1), one can give a suitable integral expression for the operator $A_0^{-\alpha}$ for $\alpha > 0$, which is bijective from $E_0 \to R(A_0^{-\alpha}) \subset E_0$; for more details see [1, III.4.6, pg. 147], [11], [12]. This implies that $A_0^\alpha = (A_0^{-\alpha})^{-1}$ is well defined, and therefore we can define

$$F_\alpha = D(A_0^\alpha) = R(A_0^{-\alpha}), \quad \alpha \geq 0 \tag{3.16}$$

with the norm $\|\cdot\|_\alpha = \|A_0^\alpha \cdot\|_0$. Note that this construction for $\alpha = n \in \mathbb{N}$ coincides with A_0^n and $F_n = E_n$.

So we get the positive fractional power scale

$$\{F_\alpha,\ \alpha \geq 0\}, \qquad A_\alpha \in \mathcal{H}(F_{\alpha+1}, F_\alpha), \qquad \alpha \geq 0, \tag{3.17}$$

where A_α is the realization of A_0 on F_α and is an isometry, see [1, V.1.2.4, pg. 258] and [1, V.1.2.6, pg. 260]. Moreover, for every $\alpha \geq 0$

$$\rho(A_\alpha) = \rho(A_0) \tag{3.18}$$

again because of [1, V.1.1.2.e), pg. 252].

For the negative scale, note that (3.15) together with (3.6) implies $(-\infty, 0] \subset \rho(A_n)$ for any $n \in \mathbb{Z}$. Fix now $N \in \mathbb{N}$ and take $A_{-N} \in \mathcal{H}(F_{-N+1}, F_{-N})$. With the construction above as in (3.16) but with the operator A_{-N} in F_{-N}, we get the extrapolated fractional power scale of order N,

$$F_{\alpha-N} = D(A_{-N}^{\alpha}) \qquad \alpha \geq 0, \tag{3.19}$$

see [1, V.1.3.8, pg. 266] and [1, V.1.3.9, pg. 267]. Then we have

$$\{F_\alpha,\ \alpha \geq -N\}, \qquad A_\alpha \in \mathcal{H}(F_{\alpha+1}, F_\alpha), \qquad \rho(A_\alpha) = \rho(A_0) \qquad \alpha \geq -N$$

and A_α is an isometry from $E_{\alpha+1}$ into E_α.

Again, $F_k = E_k$ for $k \in \mathbb{Z}$, $k \geq -N$, and for $\alpha \geq 0$, F_α and A_α above coincide with the ones in (3.17).

Now fix $A_\beta : F_{\beta+1} \to F_\beta$ for any $\beta \geq -N$. Renaming $F_\beta = Z$, $F_{\beta+1} = Z^1$ we have the following reiteration property (see [1, V.1.2.6, pg. 260] or [12, Proposition 10.6])

$$Z^\varepsilon = D(A_\beta^\varepsilon) = F_{\beta+\varepsilon} \tag{3.20}$$

for $\varepsilon \in [0, 1]$, and A_β is sectorial in Z, thus we can apply [11, I.1.4.3, pg. 26], to get

$$\|e^{-A_\beta t}\|_{\mathcal{L}(F_\beta, F_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}, \qquad t > 0, \quad \alpha \geq \beta \geq -N \tag{3.21}$$

for any $\sigma > \text{type}(A_0)$.

As above, if E_0 is reflexive, we can identify the negative side of the scale with some dual spaces by means of [1, V.1.4.12, pg. 274] getting

$$F_{-\alpha} = (F_\alpha^\sharp)' \quad \textit{and} \quad A_{-\alpha} = (A_\alpha^\sharp)', \qquad \alpha > 0 \tag{3.22}$$

with

$$e^{-A_{-\alpha} t} = (e^{-A_\alpha^\sharp t})'. \tag{3.23}$$

Therefore analogously to Definition 3.4 we say that A_0 defines an analytic semigroup $S_{A_0}(t)$ in the fractional power scale $\{F_\alpha\}_{\alpha \geq -N}$ in the sense that

$$S_{A_0}(t)|_{F_\alpha} = e^{-A_\alpha t}, \qquad \forall\ \alpha \geq -N$$

and

$$\|S_{A_0}(t)\|_{\mathcal{L}(F_\beta, F_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}, \qquad t > 0, \quad \alpha \geq \beta \geq -N.$$

Now we construct the fractional power scale and the semigroup without assuming (3.15).

Proposition 3.7 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and take c such that $(-\infty, 0] \in \rho(A_0 + cI)$.*

Then given $N \in \mathbb{N}$, the scale $\{F_\alpha\}_{\alpha \geq -N}$ generated by $A_0 + cI$, as above, is independent of c and the realizations of A_0 in F_α, denoted by A_α, satisfy

$$A_\alpha \in \mathcal{H}(F_{\alpha+1}, F_\alpha) \qquad \rho(A_\alpha) = \rho(A_0) \qquad \alpha \geq -N.$$

Hence we have an analytic semigroup $S_{A_0}(t)$ defined in the scale $\{F_\alpha\}_{\alpha\geq -N}$ such that $S_{A_0}(t)|_{F_\alpha} = e^{-A_\alpha t}$, $\alpha \geq -N$, satisfies

$$\|S_{A_0}(t)\|_{\mathcal{L}(F_\beta,F_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\sigma t} \qquad t>0, \quad \alpha\geq\beta\geq -N$$

for any $\sigma > type(A_0)$.

Furthermore if E_0 is reflexive, then $F_{-\alpha} = (F^{\sharp}_{\alpha})'$, $A_{-\alpha} = (A^{\sharp}_{\alpha})'$ and $e^{-A_{-\alpha}t} = (e^{-A^{\sharp}_{\alpha}t})'$ for $0<\alpha\leq N$.

Proof. The case $(-\infty,0] \in \rho(A_0)$ has been discussed before.

If $(-\infty,0] \notin \rho(A_0)$, there exists $c\in\mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $(-\infty,0]\in\rho(\tilde{A}_0)$. Then the corresponding scale of spaces is independent of c, see the comments on Definition 1.4.7 in [11]. Thus, with $\tilde{A}_\alpha = A_\alpha + cI$ in F_α and applying standard arguments in [13] or [11] we obtain that

$$e^{-A_\alpha t} = e^{-ct} e^{-\tilde{A}_\alpha t}$$

and the result follows. ■

Remark 3.8 *Note that after Propositions 3.6 and 3.7, for $A_0 \in \mathcal{H}(E_1,E_0)$ we have a discrete scale (3.5) and with the notations of these propositions, we have*

$$F_k = E_k \quad \textit{for } k\in\mathbb{Z},\ k\geq -N.$$

However, the intermediate spaces, F_α and E_α, for $\alpha\in\mathbb{R}\setminus\mathbb{Z}$, $\alpha\geq -N$, do not need to coincide in general. But, if A_0 has bounded imaginary powers, that is, there exist $\varepsilon>0$ and $M\geq 1$ such that

$$\|A_0^{it}\|_{\mathcal{L}(E_1,E_0)} \leq M \quad \textit{for } t\in[-\varepsilon,\varepsilon], \tag{3.24}$$

then E_α and the scale of fractional powers F_α coincide, see [1, V.1.5.13, pg. 283].

An important case when this happens is when E_0 is a Hilbert space and A_0 is selfadjoint.

Finally observe that abusing of the notations we have used the same notations A_α and $e^{-A_\alpha t}$ for both the interpolation and fractional power scales. This should produce no confusion since it will be always clear from the context what scale are we working with.

4 The scales and semigroup for A_0^2

In this section we show how the scale of spaces constructed in Section 3 for A_0 can be used for the squared operator $A_0^2 := A_0 \circ A_1$. That is, our goal here is to relate the scales of the square of an operator, A_0^2, with the scale of the A_0. We will show that if we perform the constructions in Section 3 with A_0^2 we arrive to the same spaces than for A_0 with a suitable labeling.

Hence, we assume as in the previous section that

$$A_0 \in \mathcal{H}(E_1,E_0).$$

Observe that by Propositions 3.6 and 3.7 we can consider the associated interpolation scale $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ or the fractional power scale $\{F_\alpha\}_{\alpha\geq -N}$, $N \in \mathbb{N}$ without assuming $0 \in \rho(A_0)$ or $(-\infty, 0] \in \rho(A_0)$, respectively. Also, note that with the notation of the previous section,

$$A_0^2 := A_0 \circ A_1, \qquad A_0^2 : E_2 \to E_0.$$

Hence, we will assume furthermore that

$$A_0^2 \in \mathcal{H}(E_2, E_0).$$

The following result, which is a particular case of [12, Proposition 10.5], gives a criteria for determining when A_0^2 is a sectorial operator.

Proposition 4.1 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ with $(-\infty, 0] \subset \rho(A_0)$ and satisfying $\|(A_0 - \lambda)^{-1}\| \leq \frac{K}{|\lambda|}$ for $\lambda \in S_{0,\phi}$ with $\phi \in (0, \frac{\pi}{4})$ where $S_{0,\phi}$ is a sector as (3.1) with vertex $a = 0$.*

Then A_0^2 satisfies $S_{0,2\phi} \subset \rho(A_0^2)$ and

$$\|(A_0^2 - \lambda)^{-1}\|_{E_0} \leq \frac{K}{|\lambda|}$$

for $\lambda \in S_{0,2\phi}$, thus $A_0^2 \in \mathcal{H}(E_2, E_0)$.

Remark 4.2
i) As an indication for the proof observe that to solve $A_0^2 u - \lambda u = f$, with $\lambda \in \mathbb{C}$ we can rewrite this equation as

$$(A_0 + \omega_2)(A_0 + \omega_1)u = f$$

where ω_1 and $\omega_2 = -\omega_1$ denote the complex square roots of λ. Thus λ will be in $\rho(A_0^2)$ if both ω_1, $\omega_2 \in \rho(A_0)$. In particular, if $\lambda \in S_{0,2\phi}$, with $\phi < \frac{\pi}{4}$, then ω_1, $\omega_2 \in S_{0,\phi} \subset \rho(A_0)$, thus $S_{0,2\phi} \subset \rho(A_0^2)$. For the estimate, just note that

$$\|(A_0^2 - \lambda)^{-1}\|_{E_0} \leq \|(A_0 + \omega_1)^{-1}(A_0 + \omega_2)^{-1}\|_{E_0} \leq \frac{K_1}{|\omega_1|}\|(A_0 + \omega_2)^{-1}\|_{E_0} \leq \frac{K}{|\omega_1||\omega_2|} = \frac{K}{|\lambda|}.$$

ii) $0 \in \rho(A_0)$ implies $0 \in \rho(A_0^2)$.
iii) In general, there is no relationship between type(A_0^2) and type(A_0).

So now we can construct both interpolation and fractional scales for A_0^2 following the procedures explained in Section 3. In the next two results we will show that these scales coincide with the ones for A_0 after a suitable labeling.

Proposition 4.3 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and assume $A_0^2 := A_0 \circ A_1 \in \mathcal{H}(E_2, E_0)$. Let $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ be the interpolation scale for A_0 as in Proposition 3.6. Then on the scale $X_\alpha = E_{2\alpha}$ with $\alpha \in \mathbb{R}$ we have $A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(X_{\alpha+1}, X_\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ that satisfies $S_{A_0^2}(t)|_{X_\alpha} = e^{-A_\alpha^2 t}$ and*

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(X_\beta, X_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t} \qquad t > 0, \alpha, \beta \in \mathbb{R}, \alpha \geq \beta$$

for any $\mu > type(A_0^2)$. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$X_{-\alpha} = (X_\alpha^\sharp)' \quad \textit{and} \quad A^2_{-\alpha} = (A_\alpha^{2\sharp})', \qquad \alpha > 0$$

and it holds that

$$e^{-A^2_{-\alpha}t} = (e^{-A_\alpha^{2\sharp}t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A_\alpha^2 u = 0, & t > 0 \\ u(0) = u_0 \in X_\alpha \end{cases}$$

for any $\alpha \in \mathbb{R}$ has a unique solution $u(t) = S_{A_0^2}(t)u_0 = e^{-A_\alpha^2 t}u_0$.

Proof. Step 1. We start proving the result assuming $0 \in \rho(A_0)$.

Hence, $0 \in \rho(A_0^2)$ and in this case it is easy to see that the construction (3.4)–(3.7) applied to A_0^2 leads to the discrete scale $\{X_k : k \in \mathbb{Z}\}$ with $X_k = E_{2k}$, $k \in \mathbb{Z}$ and $A_k^2 = A_k \circ A_{k+1} \in \mathcal{H}(X_{k+1}, X_k)$.

By means of the complex interpolation, the construction (3.9)–(3.12) leads for $\alpha = k+\theta$ with $\theta \in (0,1)$, $k \in \mathbb{Z}$, to

$$X_\alpha := X_{k+\theta} := [X_{k+1}, X_k]_\theta = [E_{2(k+1)}, E_{2k}]_\theta = E_{2\alpha}$$

and

$$A_\alpha^2 := A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(X_{\alpha+1}, X_\alpha)$$

for any $\alpha \in \mathbb{R}$.

In particular, by (3.12) with A_α^2, we have as in Definition 3.4 that A_0^2 defines an analytic semigroup $S_{A_0^2}(t)$ in the scale $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ that satisfies $S_{A_0^2}(t)|_{X_\alpha} = e^{-A_\alpha^2 t}$ and

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(X_\alpha, X_\beta)} \le \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t} \quad t > 0, \ \alpha, \beta \in \mathbb{R}, \alpha \ge \beta$$

for any $\mu > \mathrm{type}(A_0^2)$.

If E_0 is reflexive we can identify, as above, the negative side of this scale with some dual spaces. In fact, from (3.7) we have $X_{-k} = (X_k^\sharp)'$ and $A^2_{-k} = (A_k^{\sharp 2})'$ and by interpolation, see (3.13), $X_{-\alpha} = (X_\alpha^\sharp)'$ and $A^2_{-\alpha} = (A_\alpha^{\sharp 2})'$, $\alpha > 0$, with $e^{-A^2_{-\alpha}t} = (e^{-A_\alpha^{\sharp 2}t})'$ and $(A_\alpha^2)^\sharp = (A_\alpha^\sharp)^2$, see (3.14).

Step 2. Now, if $0 \notin \rho(A_0)$, there exists $c \in \mathbb{R}$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $0 \in \rho(\tilde{A}_0)$ and $\tilde{A}_0 \in \mathcal{H}(E_1, E_0)$. Now we prove that $\tilde{A}_0^2 \in \mathcal{H}(E_2, E_0)$. For this note that $\tilde{A}_0^2 = A_0^2 + P$, with $P = 2cA_0 + c^2 I$, which satisfies $\|P\|_{\mathcal{L}(E_1,E_0)} \le R_0$. Since $A_0^2 \in \mathcal{H}(E_2, E_0)$, using this and Corollary 1.4.5, page 27 in [11] we get $\tilde{A}_0^2 \in \mathcal{H}(E_2, E_0)$.

Therefore we can use Step 1 for $\tilde{A}_0^2$ and observe that from Proposition 3.6 the interpolation scale for $\tilde{A}_0$, $\{E_\alpha\}_{\alpha\in\mathbb{R}}$, is independent of c. Denote then $X_\alpha = E_{2\alpha}$.

Then $\tilde{A}_0^2$ defines an analytic semigroup $S_{\tilde{A}_0^2}(t)$ in the scale $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ and as above $S_{\tilde{A}_0^2}(t)|_{X_\alpha} = e^{-\tilde{A}_\alpha^2 t}$ and

$$\|S_{\tilde{A}_0^2}(t)\|_{\mathcal{L}(X_\alpha,X_\beta)} \le \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\tilde{\mu}t} \quad t>0,\ \alpha,\beta\in\mathbb{R}, \alpha\ge\beta$$

where $\tilde{\mu} > \text{type}(\tilde{A}_0^2)$.

Now we transfer this information to the semigroup defined by A_0^2. For this observe that $A_0^2 = \tilde{A}_0^2 - P$, with $P = 2cA_0 + c^2 I$ as above, and for all $\alpha\in\mathbb{R}$,

$$\|P\|_{\mathcal{L}(X_\alpha,X_{\alpha-\frac{1}{2}})} \le R_0$$

with R_0 independent of α.

Then we can apply Theorem 2.1 with $\beta = \alpha - \frac{1}{2}$ and α arbitrary, to obtain the semigroup $S_{A_0^2}(t)$ defined in X_γ for all $\gamma \in E(\alpha) := (\alpha-1, \alpha]$ and satisfying the smoothing estimate (2.7) from X_γ to $X_{\gamma'}$ for $\gamma\in E(\alpha)$ and $\gamma' \in R(\beta) := [\alpha-\frac{1}{2}, \alpha+\frac{1}{2})$, $\gamma'\ge\gamma$.

In order to extend (2.7) for all $\gamma' > \gamma$, we perform a "jump" argument as follows. Given $\alpha\in\mathbb{R}$, take $\beta = \alpha-\frac{1}{2}$ and $\alpha' > \alpha$ such that $\alpha' < \alpha + \frac{1}{2}$, so $\alpha' \in R(\beta)$. Then we can estimate the semigroup for γ' in $R(\beta')$ through an intermediate "jump", that is

$$\gamma \in E(\alpha) \to \tilde{\gamma} \in R(\beta)\cap E(\alpha') \to \gamma' \in R(\beta')$$

and using $S_{A_0^2}(t) = S_{A_0^2}(t/2)\cdot S_{A_0^2}(t/2)$

$$\|S_{A_0^2}(t)u_0\|_{\gamma'} \le \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\tilde{\gamma}}}\|S_{A_0^2}(t/2)u_0\|_{\tilde{\gamma}} \le \frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\gamma'-\tilde{\gamma}}}\frac{\tilde{M}e^{\mu(t/2)}}{(t/2)^{\tilde{\gamma}-\gamma}}\|u_0\|_\gamma = \frac{Me^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_\gamma. \quad (4.1)$$

So we get (2.7) for $\gamma\in E(\alpha) = (\alpha-1,\alpha]$ and $\gamma'\in R(\beta') = [\alpha'-\frac{1}{2}, \alpha'+\frac{1}{2})$ and M depending on γ and γ'. Iterating this process, we get (2.7) for all $\gamma'>\gamma$ with $\mu >$type(A_0^2).

For the analyticity we use Theorem 2.3. Since $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ are interpolation spaces, this scale satisfies the assumptions of case i) in Theorem 2.3; see (2.8). ■

Now we turn to the fractional power scale to obtain

Proposition 4.4 *Let $A_0\in\mathcal{H}(E_1,E_0)$ and assume $A_0^2 := A_0\circ A_1 \in \mathcal{H}(E_2,E_0)$. Let $N\in\mathbb{N}$ and $\{F_\alpha\}_{\alpha\ge -2N}$ be the fractional power scale for A_0 as in Proposition 3.7. Then on the fractional power scale $Y_\alpha = F_{2\alpha}$ with $\alpha\ge -N$ we have $A_\alpha^2 := A_\alpha\circ A_{\alpha+1}\in\mathcal{H}(Y_{\alpha+1},Y_\alpha)$ and A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{Y_\alpha\}_{\alpha\ge -N}$ that satisfies $S_{A_0^2}(t)|_{F_\alpha} = e^{-A_\alpha^2 t}$ and*

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(Y_\beta,Y_\alpha)} \le \frac{C(\alpha-\beta)}{t^{\alpha-\beta}}e^{\mu t} \qquad t>0,\ \alpha\ge\beta\ge -N$$

for any $\mu > $ type(A_0^2). The constant $C(\alpha-\beta)$ is bounded for α,β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$Y_{-\alpha} = (Y_\alpha^\sharp)' \quad \textit{and} \quad A_{-\alpha}^2 = (A_\alpha^{\sharp 2})' \qquad \alpha>0,$$

and it holds that

$$e^{-A^2_{-\alpha}t} = (e^{-A^{2\sharp}_{\alpha}t})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A^2_\alpha u = 0, & t > 0 \\ u(0) = u_0 \in Y_\alpha \end{cases}$$

for any $\alpha \geq -N$ has a unique solution $u(t) = S_{A_0^2}(t)u_0 = e^{-A^2_\alpha t}u_0$.

Proof. Step 1. We first assume that $(-\infty, 0] \subset \rho(A_0)$. As before, it is easy to see that the construction in (3.4)–(3.7) applied to A_0^2 leads to the discrete scale $\{Y_k : k \in \mathbb{Z}\}$ with $Y_k = E_{2k} = F_{2k}$, $k \in \mathbb{Z}$ and $A_k^2 = A_k \circ A_{k+1} \in \mathcal{H}(Y_{k+1}, Y_k)$.

Now for $\alpha \geq -N$ the construction in (3.19) applied to A^2_{-N}, gives a fractional power scale $\{Y_\alpha : \alpha \geq -N\}$

$$Y_\alpha = D((A^2_{-N})^{\alpha+N}),\ \alpha \geq -N, \qquad A^2_\alpha = A_\alpha \circ A_{\alpha+1} \in \mathcal{H}(Y_{\alpha+1}, Y_\alpha).$$

We prove now that $Y_\alpha = F_{2\alpha}$ for $\alpha \geq -N$. In fact, because of (3.19) and (3.20), we have

$$Y_\alpha = D((A^2_{-N})^{\alpha+N}) = D(A^{2\alpha+2N}_{-N}) = F_{2\alpha}.$$

Hence, as above, A_0^2 defines a semigroup $S_{A_0^2}(t)$ in the scale $\{Y_\alpha\}_{\alpha\geq -N}$ that satisfies $S_{A_0^2}(t)|_{F_\alpha} = e^{-A^2_\alpha t}$ and

$$\|S_{A_0^2}(t)\|_{\mathcal{L}(Y_\beta, Y_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t > 0,\ \alpha \geq \beta \geq -N$$

for any $\mu > \text{type}(A_0^2)$, see (3.21).

Also, if E_0 is reflexive we can again, by (3.22), identify the negative side of this new scale with dual spaces

$$Y_{-\alpha} = (Y^\sharp_\alpha)' \quad \textit{and} \quad A^2_{-\alpha} = (A^{\sharp 2}_\alpha)' \quad 0 < \alpha \leq N$$

and from (3.23) we get $e^{-A^2_{-\alpha}t} = (e^{-A^{\sharp 2}_\alpha t})'$.

Step 2. Now, if $(-\infty, 0] \not\subset \rho(A_0)$, there exists $c \in \rho(A_0)$ such that $\tilde{A}_0 = A_0 + cI$ satisfies $(-\infty, 0] \in \rho(\tilde{A}_0)$ and $\tilde{A}_0 \in \mathcal{H}(E_1, E_0)$. Now we prove that $\tilde{A}_0^2 \in \mathcal{H}(E_2, E_0)$. For this note that $\tilde{A}_0^2 = A_0^2 + P$, with $P = 2cA_0 + c^2 I$, which satisfies $\|P\|_{\mathcal{L}(E_1,E_0)} \leq R_0$. Since $A_0^2 \in \mathcal{H}(E_2, E_0)$, using this and Corollary 1.4.5, page 27 in [11] we get $\tilde{A}_0^2 \in \mathcal{H}(E_2, E_0)$.

Note that from Proposition 3.7 the fractional power scale for $\tilde{A}_0$ is independent of c and by Step 1 we get the fractional power scale $X_\alpha = F_{2\alpha}$ and a sectorial operator $\tilde{A}^2_\alpha = \tilde{A}_\alpha \circ \tilde{A}_{\alpha+1} \in \mathcal{H}(Y_{\alpha+1}, Y_\alpha)$. Also $\tilde{A}_0^2$ defines an analytic semigroup $S_{A_0^2}(t)$ in the scale $\{Y_\alpha\}_{\alpha\geq -N}$ and as above $S_{\tilde{A}_0^2}(t)|_{Y_\alpha} = e^{-\tilde{A}^2_\alpha t}$ and

$$\|S_{\tilde{A}_0^2}(t)\|_{\mathcal{L}(Y_\alpha, Y_\beta)} \leq \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\tilde{\mu} t}, \qquad t > 0, \quad \alpha \geq \beta \geq -N$$

where $\tilde{\mu} > \text{type}(\tilde{A}_0^2)$.

To transfer this information to the semigroup defined by A_0^2, observe that $A_0^2 = \tilde{A}_0^2 - P$ with $P = 2cA_0 + c^2 I$, as above and

$$\|P\|_{\mathcal{L}(Y_\alpha, Y_{\alpha-\frac{1}{2}})} \le R_0, \qquad \alpha \ge -N$$

with R_0 independent of α. Then, we can apply Theorem 2.1 to obtain the semigroup $S_{A_0^2}(t)$ in Y_γ and smoothing from Y_γ to $Y_{\gamma'}$ for $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ and $\gamma' \in R(\beta) := [\alpha - \frac{1}{2}, \alpha + \frac{1}{2})$, $\gamma' \ge \gamma$. A similar jump argument as (4.1) concludes the estimate for all $\gamma' > \gamma \ge -N$.

Finally, the analyticity comes again from Theorem 2.3, part ii). In fact note that fractional power spaces satisfty (2.9), see [1, V.(1.2.12)]. ■

Remark 4.5 *According to Remark 3.8 if A_0 has bounded imaginary powers, then A_0^2 does as well, see (3.24). In such case both scales and semigroups in Propositions 4.3 and 4.4 coincide, that is, $X_\alpha = Y_\alpha$ for $\alpha \ge -N$, see [1, V.1.5.13, pg. 283].*

5 Some fourth order equations in the Bessel-Lebesgue spaces in $\mathbb{R}^N$

We will apply the results in Section 4 to prove that the bi-Laplacian in some scales of spaces defines an analytic semigroup and the bi-Laplacian equation (5.2) has a unique solution. Then we will consider a general class of perturbations, namely derivative operators, even with space dependence, to which we will apply the results in Section 2 so that the perturbed bi-Laplacian equation will be well possed.

We take, $A_0 = -\Delta$ in $L^q(\mathbb{R}^N)$, with $1 < q < \infty$ with domain $D(A_0) = H^{2,q}(\mathbb{R}^N)$, where $H^{k,q}(\mathbb{R}^N)$, $k \in \mathbb{N}$ denotes the standard Sobolev spaces (often denoted $W^{k,q}(\mathbb{R}^N)$). In this setting, $-\Delta$ is a sectorial operator, [11], [2]. Even more using [2, 9.7, pg. 648] we get that $-\Delta$ (and therefore Δ^2 by Remark 4.5) has bounded imaginary powers in $L^q(\mathbb{R}^N)$ for $1 < q < \infty$. Hence, in the following examples the fractional power scale and the complex interpolation scale of Section 3 will coincide.

Note that $L^q(\mathbb{R}^N)$ is reflexive so that the negative scale is described as dual spaces, see Section 3.

Using the complex interpolation/extrapolation scale with $E_0 = L^q(\mathbb{R}^N)$ and $E_1 = H^{2,q}(\mathbb{R}^N)$ as in Section 3.1 leads to the scale of Bessel spaces. These spaces are very convenient because they satisfy the sharp Sobolev embeddings

$$H^{s,q}(\mathbb{R}^N) \subset \begin{cases} L^r(\mathbb{R}^N),\ s - \frac{N}{q} \ge -\frac{N}{r},\ 1 \le r < \infty & if\ s - \frac{N}{q} < 0 \\ L^r(\mathbb{R}^N),\ 1 \le r < \infty & if\ s - \frac{N}{q} = 0 \\ C^\eta(\mathbb{R}^N) & if\ s - \frac{N}{q} > \eta \ge 0. \end{cases}$$

For more details, see [11, pg. 35], [1, I.2] or [16, 1]. In what follows we will denote $E_\alpha := H^{2\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$.

Therefore for $1 < q < \infty$ the heat equation

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0, & \text{in } \mathbb{R}^N \end{cases} \tag{5.1}$$

defines a semigroup $S_{-\Delta}(t)$ in the scale of Bessel spaces $\{E_\alpha\}_{\alpha\in\mathbb{R}}$ that satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N)$$

for $1 \leq q \leq r \leq \infty$ and some constant $M_{r,q}$ and

$$\|S_{-\Delta}(t)u_0\|_{H^{2\alpha,q}(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in H^{2\beta,q}(\mathbb{R}^N)$$

for $1 < q < \infty$, $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \beta$. In both estimates above $\mu_0 > 0$ can be arbitrarily small, because type$(-\Delta) = 0$. This as well as some other useful properties of $-\Delta$ and Δ^2 in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, are collected in the next Lemma.

Lemma 5.1 *Take $1 < q < \infty$ and denote $E_0 = L^q(\mathbb{R}^N)$.*
i) The Laplace operator $-\Delta$ in E_0 with domain $E_1 = D(-\Delta) = H^{2,q}(\mathbb{R}^N)$ satisfies the estimate

$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leq M|\lambda|^{-1} \qquad \text{for all } \lambda \in S_{0,\phi}$$

for $S_{0,\phi}$ as in (3.1), $\phi > 0$ arbitrarily small. Furthermore $\sigma(-\Delta) = [0, \infty)$ and therefore

$$\text{type}(-\Delta) = 0.$$

ii) The bi–Laplacian operator Δ^2 in E_0 with domain $E_2 = D(\Delta^2) = H^{4,q}(\mathbb{R}^N)$ satisfies the estimate

$$\|(\Delta^2 - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leq M|\lambda|^{-1} \qquad \text{for all } \lambda \in S_{0,2\phi}$$

with $\phi > 0$ arbitrarily small. Furthermore $\sigma(\Delta^2) = [0, \infty)$ and therefore

$$\text{type}(\Delta^2) = 0.$$

Proof. The first part, for the Laplacian, is well known. The resolvent estimate, in particular, can be found in pages 32 and 33 of [11].

For proving ii), since in i) $\phi > 0$ can be taken arbitrarily small, we can apply Proposition 4.1 and we get that Δ^2 is sectorial with sector $S_{0,2\phi}$, where $2\phi > 0$ can be arbitrarily small. Then $\sigma(\Delta^2) \subset [0, \infty)$ is an immediate consequence of the fact that $\phi > 0$ is arbitrarily small. On the other hand, as we will show in Proposition 6.3, working in the uniform space $\dot{L}^q_U(\mathbb{R}^N)$ we actually have $\sigma(\Delta^2) = [0, \infty)$. Then, using [1, Lemma V.1.1.1, pg. 250] we get $\sigma(\Delta^2) = [0, \infty)$ in $L^q(\mathbb{R}^N)$ as well. From this, we get type$(\Delta^2) = 0$. ■

Now, we can apply Proposition 4.3 to get

Lemma 5.2 *Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0, & in\ \mathbb{R}^N. \end{cases} \tag{5.2}$$

i) Then for each $1 < q < \infty$, *(5.2) defines an analytic semigroup,* $S_{\Delta^2}(t)$, *in the scale* $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, *such that for any* $\mu_0 > 0$ *there exists* C *such that*

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(H^{4\beta,q}(\mathbb{R}^N),H^{4\alpha,q}(\mathbb{R}^N))} \le \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \ge \beta.$$

ii) The analytic semigroup $S_{\Delta^2}(t)$, *in* $L^q(\mathbb{R}^N)$, $1 < q < \infty$, *satisfies*

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N),L^r(\mathbb{R}^N))} \le \frac{M_{q,r}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu_0 t} \qquad t > 0$$

for any $\mu_0 > 0$ *and* $1 < q \le r \le \infty$ *and some* $M_{q,r} > 0$.

Proof.
i) We use Proposition 4.3 for $A_0 = -\Delta$ (note that if suffices to take $c = 1$ in the proof of the proposition), and we get that $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$.

Note that from Lemma 5.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary.

ii) For $1 < q < \infty$, we use i) with $\alpha = 0$ and we have that Δ^2 defines an analytic semigroup in $L^q(\mathbb{R}^N)$.

Now, if $r \ge q$ we use i) again, now with $\beta = 0$, and choosing α such that

$$-\frac{N}{r} = 4\alpha - \frac{N}{q}$$

and we get

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \le \|S_{\Delta^2}(t)u_0\|_{H^{4\alpha,q}(\mathbb{R}^N)} \le \frac{M_\alpha e^{\mu_0 t}}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^N)},$$

which leads to

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \le \frac{M_{r,q} e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}.$$

Again, because of part ii) of Lemma 5.1, type$(\Delta^2) = 0$ and then $\mu_0 > 0$ is arbitrary.
■

Remark 5.3 *For* $q = 1$, *if we take any* $r > 1$ *and* $\beta > \frac{N}{4r'}$ *then we have* $H^{4\beta,r'}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *and therefore* $L^1(\mathbb{R}^N) \hookrightarrow H^{-4\beta,r}(\mathbb{R}^N)$.

Now using i) with $\alpha = 0$ *we get*

$$\|S_{\Delta^2}(t)u_0\|_{L^r(\mathbb{R}^N)} \le \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{H^{-4\beta,r}(\mathbb{R}^N)} \le \frac{M_{r,1} e^{\mu_0 t}}{t^\beta} \|u_0\|_{L^1(\mathbb{R}^N)}$$

for any $\beta > \frac{N}{4}(1 - \frac{1}{r})$. *Hence we obtain the estimate in ii) for* $q = 1$ *and any* $r > 1$, *for an exponent as close as we want to* $\frac{N}{4}(1 - \frac{1}{r})$.

Observe that the solution of problem (5.2) can be described as the convolution of the initial data with the self-similar kernel for the bi–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [9, 10] and [8, 6].

Now we can use the results in Section 2 to perturb equation (5.2). For this we first consider perturbations which do not involve derivatives and that can be handled with the semigroup defined by (5.2) in the scale of Lebesgue spaces as in part ii) in Lemma 5.2. For this, as a consequence of Hölder inequality, we get a result, as in [14, Lemma 21, pg. 37].

Lemma 5.4 *Assume that* $m \in L^p(\mathbb{R}^N)$, *then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geq p'$ *and* $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, *that*

$$P \in \mathcal{L}(L^r(\mathbb{R}^N), L^s(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(L^r(\mathbb{R}^N), L^s(\mathbb{R}^N))} \leq C\|m\|_{L^p(\mathbb{R}^N)}.$$

Then we obtain the following preliminary result. This will be later extended to Bessel spaces, see Theorem 5.10 below.

Lemma 5.5 *Let* m *be such that* $\|m\|_{L^p(\mathbb{R}^N)} \leq R_0$, *with* $p > \frac{N}{4}$. *Then for any* $1 < q < \infty$ *the problem*

$$\begin{cases} u_t + \Delta^2 u = m(x)u & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases}$$

defines an analytic semigroup $S(t)$ *in* $L^q(\mathbb{R}^N)$ *that satisfies*

$$\|S(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in L^q(\mathbb{R}^N)$$

for $1 < q \leq r \leq \infty$ *with* $M_{q,r}$ *and* μ *depending on* m *only through* R_0.

Furthermore, if, as $\varepsilon \to 0$,

$$m_\varepsilon \to m \quad \text{in } L^p(\mathbb{R}^N), \quad p > \frac{N}{4}$$

then for every $1 < q \leq r \leq \infty$ *and* $T > 0$ *there exists* $C(\varepsilon) \to 0$ *as* $\varepsilon \to 0$, *such that the corresponding semigroups satisfy*

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Proof. We denote $Z_{\alpha(r)} := L^r(\mathbb{R}^N)$, $\alpha(r) = -\frac{N}{4r} \in I := [-\frac{N}{4}, 0]$, note that this scale is not nested. From Lemma 5.2 ii) we get that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(Z_\beta, Z_\alpha)} \leq \frac{C}{t^{\alpha-\beta}}, \qquad 0 < t \leq 1, \quad \alpha \geq \beta,$$

for any $\alpha, \beta \in I = [-\frac{N}{4}, 0]$. On the other hand, from Lemma 5.4, we have for $r \geq p'$, that is, for each $\alpha_0 := -\frac{N}{4p'} \leq \alpha \leq 0$

$$P \in \mathcal{L}(Z_\alpha, Z_\beta), \quad \|P\|_{\mathcal{L}(Z_\alpha, Z_\beta)} \leq C\|m\|_{L^p(\mathbb{R}^N)}$$

with $\alpha = -\frac{N}{4r}$, $\beta = -\frac{N}{4s} = \alpha - \frac{N}{4p}$ and $0 \leq \alpha - \beta = \frac{N}{4p} < 1$, since $p > \frac{N}{4}$.

Hence, with α and β fixed as above, we can apply Theorem 2.1 and we get a semigroup $S(t) = S_P(t)$ in Z_γ for $\gamma \in [\beta, \alpha]$ and satisfying the smoothing estimates

$$\|S(t)\|_{\mathcal{L}(Z_\gamma, Z_{\gamma'})} \leq \frac{M_{\gamma,\gamma'} e^{\omega t}}{t^{\gamma' - \gamma}}$$

for the indexes

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha] \cap I, \qquad \gamma' \in R(\beta) = [\beta, \beta + 1) \cap I, \qquad \gamma' \geq \gamma.$$

Now we show that as α ranges in $[\alpha_0, 0]$, γ, γ' range in $I = [-\frac{N}{4}, 0]$. To see this, recall that $\alpha \in [\alpha_0, 0]$, $\beta = \alpha - \frac{N}{4p}$ and we can take $\gamma, \gamma' \in [\beta, \alpha]$. Thus using a "jump" argument as in (4.1) we just need to find the smallest β and the biggest α. Since $\beta = \alpha - \frac{N}{4p}$ the smallest β is $\beta = \alpha_0 - \frac{N}{4p} = -\frac{N}{4}$, while the biggest α is $\alpha = 0$.

For the convergence of the semigroups, first, using Lemma 5.4 we get that $\|P_\varepsilon - P\|_{\mathcal{L}(L^r(\mathbb{R}^N), L^s(\mathbb{R}^N))} \to 0$, that is $\|P_\varepsilon - P\|_{\mathcal{L}(Z_\alpha, Z_\beta)} \to 0$ for any $\alpha \in [\alpha_0, 0]$, $\beta = \alpha - \frac{N}{p}$. Now we can apply Theorem 2.2 to get the convergence of the semigroup.

The analyticity will follow from Theorem 5.10 below for $a = b = 0$. ■

Remark 5.6 *For a similar result with $q = 1$, see Remark 5.3.*

We are now going to work with more general perturbations and in particular we will consider perturbations that involve derivatives. For this we will need to work with the semigroup defined by (5.2) in the scale of Bessel spaces as in part i) of Lemma 5.2. For this, let D^r denote any partial derivative of order $r \in \mathbb{N}$ and fix $m \in \mathbb{N}$.

Then if $m \geq r$, we have $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$. On the other hand, $D^r : H^{-m,q}(\mathbb{R}^N) \to H^{-m-r,q}(\mathbb{R}^N)$, is defined as

$$< D^r u, \varphi >= (-1)^r \int_{\mathbb{R}^N} u D^r \varphi, \quad \text{for all } \varphi \in H^{m+r,q'}(\mathbb{R}^N).$$

Finally, if $m < r$, $D^r : H^{m,q}(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$ is defined as

$$< D^r u, \varphi >= (-1)^{r-m} \int_{\mathbb{R}^N} D^m u D^{r-m} \varphi, \quad \text{for all } \varphi \in H^{r-m,q'}(\mathbb{R}^N)$$

which corresponds to the composition $D^r = D^{r-m} D^m$, where $D^m : H^{m,q}(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ and $D^{r-m} : L^q(\mathbb{R}^N) \to H^{m-r,q}(\mathbb{R}^N)$.

Thus for any $1 < q < \infty$, $r \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have

$$D^r \in \mathcal{L}(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)), \quad \|D^r\|_{\mathcal{L}(H^{m,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N))} \leq C$$

for some C independent of r, m, q.

Now we extend this definition to non-integer m. For this take $m \in \mathbb{Z}$ and $s \in (m, m+1)$ and take $\theta \in (0,1)$ such that $s = \theta m + (1-\theta)(m+1)$.

Then by interpolation

$$D^r : [H^{m+1,q}(\mathbb{R}^N), H^{m,q}(\mathbb{R}^N)]_\theta = H^{s,q}(\mathbb{R}^N) \to [H^{m+1-r,q}(\mathbb{R}^N), H^{m-r,q}(\mathbb{R}^N)]_\theta = H^{s-r,q}(\mathbb{R}^N),$$

and we get that for any $r \in \mathbb{N}$ and $s \in \mathbb{R}$

$$D^r \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N)), \quad \|D^r\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N))} \leq C \tag{5.3}$$

for some C independent of r, s, q.

Using this and the results in Section 2 we get the following result in which we allow perturbations with derivatives of order $k \leq 3$.

Lemma 5.7 *Take $J \in \mathbb{N}$ and $a_j \in \mathbb{R}$, $k_j \in \mathbb{N}$ for $j = 1, ..., J$ with $\max_j |a_j| \leq R_0$ and $k = \max_j |k_j| \leq 3$. Then for each $1 < q < \infty$ the problem*

$$\begin{cases} u_t + \Delta^2 u + \sum_{j=0}^J a_j D^{k_j} u = 0, & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \textit{in } \mathbb{R}^N \end{cases}$$

defines an analytic semigroup, $S(t)$, on the scale $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, for any $\alpha \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathcal{L}(H^{4\beta,q}(\mathbb{R}^N), H^{4\alpha,q}(\mathbb{R}^N))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta$$

and also

$$\|S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(q,r)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} e^{\mu t} \qquad t > 0,$$

for $1 < q \leq r \leq \infty$, with μ, $C(\alpha-\beta)$, $C(q,r)$ depending on $\{a_j\}$ only through R_0. The constant $C(\alpha-\beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

Furthermore, if for all $j = 1, ..., J$, we have $a_j^\varepsilon \to a_j$ as $\varepsilon \to 0$ then for any $T > 0$, $\alpha \geq \beta$ or $r \geq q$, there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that the corresponding semigroups satisfy

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(H^{4\beta,q}(\mathbb{R}^N), H^{4\alpha,q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\alpha-\beta}}, \qquad \forall\, 0 < t \leq T$$

and

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T$$

for $1 < q \leq r \leq \infty$.

Proof. Since $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, we get from Lemma 5.2 i) that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(X_\beta, X_\alpha)} \leq \frac{C}{t^{\alpha-\beta}}, \qquad 0 < t \leq 1,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta.$$

From (5.3) each of the perturbations $P_j = a_j D^{k_j}$ satisfies $\|P_j\|_{\mathcal{L}(X_\alpha, X_{\alpha-k_j/4})} \leq C$ for all $\alpha \in \mathbb{R}$ with $C = C(R_0)$ independent of j, and we have that

$$P = \sum_j^J P_j \in \mathcal{L}(X_\alpha, X_{\alpha-k/4}), \qquad \|P\|_{\mathcal{L}(X_\alpha, X_{\alpha-k/4})} \leq C(J, R_0).$$

Hence, we can apply Theorem 2.1 with $\alpha \in \mathbb{R}$, $\beta = \alpha - \frac{k}{4}$ and since the scale is nested, we get a semigroup $S(t) = S_P(t)$ in X_γ for $\gamma \in E(\alpha) := (\alpha - 1, \alpha]$ that satisfies the smoothing estimates

$$\|S(t)\|_{\mathcal{L}(X_\gamma, X_{\gamma'})} \leq \frac{M_{\gamma,\gamma'} e^{\mu t}}{t^{\gamma'-\gamma}}$$

for every γ, γ' such that

$$\gamma \in E(\alpha) := (\alpha - 1, \alpha] \qquad \text{and} \qquad \gamma' \in R(\beta) := [\alpha - k/4, \alpha + k/4), \qquad \gamma' \geq \gamma.$$

Again, since $\alpha \in \mathbb{R}$ is arbitrary we can use the “jump” argument as in (4.1), we get the smoothing estimate for any $\gamma, \gamma' \in \mathbb{R}$, $\gamma' > \gamma$.

The analyticity comes again from Lemma 5.2 and part i) in Theorem 2.3.

Now, if $1 < q < \infty$ and $r \geq q$ we take $\beta = 0$ and α such that $H^{4\alpha,q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, that is $-\frac{N}{r} = 4\alpha - \frac{N}{q}$. Then we get

$$\|S(t)u_0\|_{L^r(\mathbb{R}^N)} \leq C\|S(t)u_0\|_{H^{4\alpha,q}(\mathbb{R}^N)} \leq \frac{C(\alpha)e^{\mu t}}{t^\alpha}\|u_0\|_{L^q(\mathbb{R}^N)} = \frac{C_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}$$

The convergence of the semigroups is consequence of Theorem 2.2 since if $a_j^\varepsilon \to a_j$ we would have $P_\varepsilon \to P$ in $\mathcal{L}(X_\alpha, X_{\alpha-k/4})$ as $\varepsilon \to 0$ for any α. ■

Remark 5.8 *For a similar result with $q = 1$, see Remark 5.3.*

Finally, we study more general perturbations in which we allow a space dependence. For this, take $k \in \mathbb{N}$ which is the order of the perturbation and take $a, b \in \mathbb{N}$ such that $a + b = k$. We define $P_{a,b}$ to be a perturbation of the form

$$P_{a,b}u = D^b(d(x)D^a u) \qquad x \in \mathbb{R}^N$$

for a given function $d(x)$ with $x \in \mathbb{R}^N$, in the sense that for any smooth enough φ

$$< P_{a,b}u, \varphi >= (-1)^b \int_{\mathbb{R}^N} d(x)D^a u D^b \varphi. \tag{5.4}$$

We will assume below that the coefficient $d(x)$ belongs to the locally uniform space $L^p_U(\mathbb{R}^N)$ composed of the functions $f \in L^p_{loc}(\mathbb{R}^N)$ such that there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^N$

$$\int_{B(x_0,1)} |f|^p \leq C \tag{5.5}$$

endowed with the norm

$$\|f\|_{L^p_U(\mathbb{R}^N)} = \sup_{x_0\in\mathbb{R}^N} \|f\|_{L^p(B(x_0,1))}.$$

The following result states the spaces of the Bessel scale between which a perturbation $P_{a,b}$ is a well behaved linear operator.

Proposition 5.9 *Let $P_{a,b}$ be as above, $d \in L^p_U(\mathbb{R}^N)$ and let $s \geq a$, $\sigma \geq b$. Then for $1 < q < \infty$ and*

$$(s - a - \frac{N}{q})_- + (\sigma - b - \frac{N}{q'})_- \geq -\frac{N}{p'} \tag{5.6}$$

we have

$$P_{a,b} \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{-\sigma,q}(\mathbb{R}^N)), \qquad \|P_{a,b}\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N),H^{-\sigma,q}(\mathbb{R}^N))} \leq C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Proof. Let $\{Q_i\}$, $i \in \mathbb{Z}^N$ be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with sides of length 1, parallel to the axes. Note that $\mathbb{R}^N = \cup_{i\in\mathbb{Z}^N}\overline{Q_i}$ and $Q_i \cap Q_j = \emptyset$ for $i \neq j$. Then

$$|\int_{\mathbb{R}^N} dD^a u D^b\varphi| \leq \sum_i |\int_{Q_i} dD^a u D^b\varphi| \leq \sum_i (\int_{Q_i} |d|^p)^{\frac{1}{p}} (\int_{Q_i} |D^a u|^n)^{\frac{1}{n}} (\int_{Q_i} |D^b\varphi|^\tau)^{\frac{1}{\tau}}$$

where we have applied Hölder's inequality with $\frac{1}{p} + \frac{1}{n} + \frac{1}{\tau} = 1$. If (5.6) holds, we can choose n, τ as before such that $s - \frac{N}{q} \geq a - \frac{N}{n}$ and $\sigma - \frac{N}{q'} \geq b - \frac{N}{\tau}$. Now, we can use the embeddings of Bessel spaces and, for some C is independent of the cube Q_i, obtain

$$\begin{aligned} |\int_{\mathbb{R}^N} dD^a u D^b\varphi| &\leq C\|d\|_{L^p_U(\mathbb{R}^N)} \sum_i \|u\|_{H^{s,q}(Q_i)} \|\varphi\|_{H^{\sigma,q'}(Q_i)} \\ &\leq C\|d\|_{L^p_U(\mathbb{R}^N)} \left(\sum_i \|u\|^q_{H^{s,q}(Q_i)}\right)^{1/q} \left(\sum_i \|\varphi\|^{q'}_{H^{\sigma,q'}(Q_i)}\right)^{1/q'}. \end{aligned} \tag{5.7}$$

Then, as in [4, Lemma 2.4], we get for any $0 \leq \alpha \leq 2$ and any $1 < q < \infty$

$$\sum_i \|\phi\|^q_{H^{2\alpha,q}(Q_i)} \leq C\|\phi\|^q_{H^{2\alpha,q}(\mathbb{R}^N)} \qquad \text{for all} \quad \phi \in H^{2\alpha,q}(\mathbb{R}^N),$$

and we obtain from (5.7)

$$|\int_{\mathbb{R}^N} dD^a u^b\varphi| \leq C\|d\|_{L^p_U(\mathbb{R}^N)} \|u\|_{H^{s,q}(\mathbb{R}^N)} \|\varphi\|_{H^{\sigma,q'}(\mathbb{R}^N)}$$

which gives the result. ■

Now we can use again the results in Section 2 to obtain the following.

Theorem 5.10 *Let $P_{a,b}$ be as in (5.4) with $k,a,b \in \{0,1,2,3\}$, $k = a+b$. Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \le R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q,a,b) \subset (-1+\frac{a}{4}, 1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma \in I(q,a,b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{4\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \le \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a,b)$ with $\gamma' \ge \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \le \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \le r \le \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q,a,b)$ is given by

$$I(q,a,b) = (-1+\frac{a}{4}+\frac{N}{4}(\frac{1}{p}-\frac{1}{q'})_+, 1-\frac{b}{4}-\frac{N}{4}(\frac{1}{p}-\frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \le T$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma \ge \gamma'$ and for any $1 < q \le r \le \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \le T.$$

Proof. By Proposition 5.9 and using $X_\alpha = E_{2\alpha} = H^{4\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, if we assume for a moment that (5.6) is satisfied for some s and σ, then it would be true that

$$P \in \mathcal{L}(X_{s/4}, X_{-\sigma/4}), \qquad \|P\|_{\mathcal{L}(X_{s/4}, X_{-\sigma/4})} \le C\|d\|_{L^p_U(\mathbb{R}^N)}.$$

Hence we can apply Theorem 2.1 above with $\alpha = s/4$ and $\beta = \sigma/4$ provided $0 \le \alpha - \beta < 1$, that is $s + \sigma < 4$.

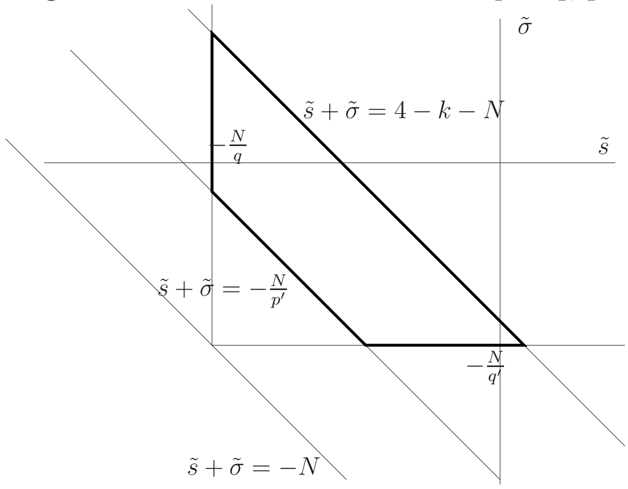
Thus, we check now that (5.6) and $s+\sigma < 4$ hold for suitable pairs (s,σ). For this we rewrite the ranges for s, σ in Proposition 5.9 in terms of $\tilde{s} = s - a - \frac{N}{q}$ and $\tilde{\sigma} = \sigma - b - \frac{N}{q'}$, so $\tilde{s} \geq -\frac{N}{q}, \tilde{\sigma} \geq -\frac{N}{q'}$ since $s \geq a, \sigma \geq b$. Then (5.6) and $s+\sigma < 4$ read

$$\tilde{s} \geq -\frac{N}{q},\ \tilde{\sigma} \geq -\frac{N}{q'}, \qquad -\frac{N}{p'} \leq \tilde{s}_- + \tilde{\sigma}_-, \qquad \tilde{s} + \tilde{\sigma} < 4 - k - N. \tag{5.8}$$

Note that since necessarily $-\frac{N}{p'} < 4 - k - N$, we get that $p > \frac{N}{4-k}$.

The set of admissible parameters $(\tilde{s}, \tilde{\sigma})$ given by (5.8) depends on the relationship between q, q' and p. Note that (5.8) defines a planar trapezium–shaped polygon, $\tilde{\mathcal{P}}$, whose long base is on the line $\tilde{s} + \tilde{\sigma} = 4 - k - N$ and the short base is on the line $\tilde{s} + \tilde{\sigma} = -\frac{N}{p'}$ in the third quadrant. As for the lateral sides note that the restriction $-\frac{N}{p'} \leq \tilde{s}_- + \tilde{\sigma}_-$ adds the condition that $\tilde{s} \geq -\frac{N}{p'}$ in the second quadrant and $\tilde{\sigma} \geq -\frac{N}{p'}$ in the fourth. These have to be combined with $\tilde{s} \geq -\frac{N}{q}$ and $\tilde{\sigma} \geq -\frac{N}{q'}$. Therefore the lateral sides are given by the lines $\tilde{s} = \max\{-\frac{N}{p'}, -\frac{N}{q}\}$ and $\tilde{\sigma} = \max\{-\frac{N}{p'}, -\frac{N}{q'}\}$. One of the possible cases is depicted in Figure 1.

Figure 1: Admissible $\tilde{s}$ and $\tilde{\sigma}$ with $p > q, q'$



Note that the polygon $\tilde{\mathcal{P}}$ transforms into a similar shaped polygon $\mathcal{P}$ which determines the region of admissible pairs (s,σ).

In any case, projecting $\tilde{\mathcal{P}}$ onto the axes gives the following ranges for $\tilde{s}$ and $\tilde{\sigma}$

$$\tilde{s} \in [\max\{-\frac{N}{p'}, -\frac{N}{q}\}, 4 - k - N - \max\{-\frac{N}{p'}, -\frac{N}{q'}\})$$

$$\tilde{\sigma} \in [\max\{-\frac{N}{p'}, -\frac{N}{q'}\}, 4 - k - N - \max\{-\frac{N}{p'}, -\frac{N}{q}\}).$$

Thus

$$s \in J_1 = [a + (\frac{N}{q} - \frac{N}{p'})_+, 4 - b - (\frac{N}{q'} - \frac{N}{p'})_+)$$

$$\sigma \in J_2 = [b + (\frac{N}{q'} - \frac{N}{p'})_+, 4 - a - (\frac{N}{q} - \frac{N}{p'})_+).$$

For each pair of admissible pairs $(s, \sigma) \in \mathcal{P}$, by Theorem 2.1 with $\alpha = \frac{s}{4}$ and $\beta = \frac{\sigma}{4}$, we get a perturbed semigroup and smoothing estimates (2.7) in the spaces corresponding to γ and γ' as in (2.6), i.e.

$$\gamma \in E(\alpha) = (\alpha - 1, \alpha], \qquad \gamma' \in R(\beta) = [\beta, \beta + 1), \qquad \gamma' \geq \gamma.$$

Hence as (s, σ) range in the region $\mathcal{P}$ a repeated "jump" argument as in (4.1) gives that the smoothing estimates hold for $\gamma \in \bigcup_{(s,\sigma)\in\mathcal{P}} E(s/4)$ and $\gamma' \in \bigcup_{(s,\sigma)\in\mathcal{P}} R(\sigma/4)$, $\gamma' \geq \gamma$. This leads to

$$\gamma \in (\frac{\inf J_1}{4} - 1, \frac{\sup J_1}{4}], \qquad \gamma' \in [-\frac{\sup J_2}{4}, 1 - \frac{\inf J_2}{4}), \gamma' \geq \gamma$$

which, after a simple calculation, reads

$$\gamma, \gamma' \in I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{q'} - \frac{1}{p'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{q} - \frac{1}{p'})_+).$$

For the estimates in Lebesgue spaces we use the Sobolev inclusions. Taking $1 < q < \infty$, $\gamma = 0$ and $0 < \gamma' \in I(q, a, b)$ we define $r > q$ such that $H^{4\gamma',q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$, that is $-\frac{N}{r} = 4\gamma' - \frac{N}{q}$. Then we get

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma'}e^{\mu t}}{t^{\gamma'}}\|u_0\|_{L^q(\mathbb{R}^N)}$$

and $\gamma' = \frac{N}{4}(\frac{1}{q} - \frac{1}{r})$. Now we follow a jump argument as in (4.1) where we take $S_{P_{a,b}}(t/2)u_0$ as initial data in $L^r(\mathbb{R}^N)$, repeat the argument above to estimate $S_{P_{a,b}}(t)u_0$ in $L^{\tilde{r}}(\mathbb{R}^N)$ for $\tilde{r} > r > q$. Since the intervals $I(r, a, b)$ contain $(-1 + \frac{a}{4} - \frac{N}{4p}, 1 - \frac{b}{4} - \frac{N}{4p})$ which do not depend on r, repeating the jump process several times we can get the estimate for any $\tilde{r} \geq q$.

The convergence of the semigroups is a direct consequence of Theorem 2.2, since Proposition 5.9 gives that if $d_\varepsilon \to d$ in $L^p_U(\mathbb{R}^N)$, then $P_\varepsilon \to P$ in $\mathcal{L}(X_{s/4}, X_{\sigma/4})$ for any pair of admissible $(s, \sigma) \in \mathcal{P}$. The case of Lebesgue spaces follows from this as well.

Finally, the analyticity comes again from Lemma 5.2 and part i) in Theorem 2.3. ■

Remark 5.11 *Note that different perturbations $P_{a,b}$ can be combined together, although not all combinations are allowed.*

In fact, if we consider two such perturbations, say $P_{a,b}$ and $P_{c,d}$, then they can be combined provided

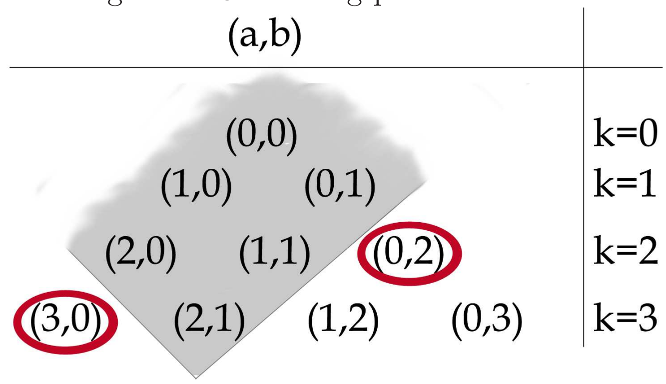
$$\max\{a, c\} + \max\{b, d\} < 4$$

with an interval for $P = P_{a,b} + P_{c,d}$ given by $I(q, P) := I(q, \max\{a, c\}, \max\{b, d\})$. Observe that there are 127 possible such combinations.

Here we present a scheme for determining, given a perturbation $P_{a,b}$, which are the other ones allowed to be combined with it. For example, if we fix a perturbation $P_{a,b}$ with $k=3$, then, all perturbations $P_{c,d}$ with $c \leq a$ and $d \leq b$ can be combined with it, and the interval is $I(q,P) = I(q,a,b)$.

For example a perturbation $P_{2,1}$ can be combined with all the ones included in the shaded area in Figure 2 with interval $I(q,2,1)$. However, the encircled perturbations $P_{3,0}$ and $P_{0,2}$ cannot be combined together.

Figure 2: Combining perturbations.



If we fix a perturbation $P_{a,b}$ with $k=2$ then, all perturbations $P_{c,d}$ with $c \leq a$ and $d \leq b$ can be combined with it, and also those with $c-1 \leq a$ or $d-1 \leq b$, but not both at the same time.

The same happens for $P_{a,b}$ with $k=1$, all perturbations $P_{c,d}$ with $k \leq 1$ can be combined with it.

Observe that perturbations in (5.4) can be handled as above because we could determine the spaces of the Bessel scale between which a perturbation $P_{a,b}$ is a well behaved linear operator; see Proposition 5.9. However the fact that a, b are integer derivatives is not really essential. Therefore, this class of perturbations can be extended to the following one, where derivatives are replaced by fractional powers of the Laplacian as long as this one is well defined in our scale. For example $-\Delta + cI$, with $c > 0$ can be used in this way, because the operator $(-\Delta + cI)^{r/2}$, $r > 0$ satisfies for any $s \in \mathbb{R}$,

$$(-\Delta + cI)^{r/2} \in \mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N)), \quad \|(-\Delta + cI)^{r/2}\|_{\mathcal{L}(H^{s,q}(\mathbb{R}^N), H^{s-r,q}(\mathbb{R}^N))} \leq C$$

for some C independent of s, r, q. Note that this estimate is analogous to (5.3) for a non-integer r.

Thus, the perturbations

$$P_{a,b}u = (-\Delta + cI)^{b/2}(d(x)(-\Delta + cI)^{a/2}u) \quad a, b \geq 0$$

for any a, $b \in \mathbb{R}$, in the sense that for any smooth enough φ

$$< P_{a,b}u, \varphi >= \int_{\mathbb{R}^N} d(x)(-\Delta + cI)^{a/2}u(-\Delta + cI)^{b/2}\varphi, \tag{5.9}$$

with $d \in L^p_U(\mathbb{R}^N)$, satisfy the statement in Proposition 5.9.

Then proceeding exactly as in Theorem 5.10, we recover the same results for this kind of perturbations, with the only difference that now $k = a + b$ is a real number smaller than 4.

Theorem 5.12 *Let $a, b, k \geq 0$ be real numbers such that $k = a + b < 4$ and $P_{a,b}$ be as in (5.9). Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-k}$, then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q, a, b) \subset (-1+\frac{a}{4}, 1-\frac{b}{4})$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{b}{4}-\frac{N}{4p})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{4\gamma,q}(\mathbb{R}^N)$, $1 < q < \infty$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + P_{a,b} u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{4\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{H^{4\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{4\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = (-1 + \frac{a}{4} + \frac{N}{4}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{4} - \frac{N}{4}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $1 < q \leq r \leq \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{4\gamma,q}(\mathbb{R}^N), H^{4\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' > \gamma$ and for any $1 < q \leq r \leq \infty$

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T.$$

Note that Remarks 5.3 and 5.11 apply here as well.

6 Fourth order equations in the uniform Bessel-Lebesgue spaces in $\mathbb{R}^N$

The heat equation (5.1) and therefore the bi-Laplacian equation (5.2) can be also considered in much larger spaces than the Bessel spaces above, by taking the initial data in locally uniform spaces.

For this consider the locally uniform space $L^q_U(\mathbb{R}^N)$ for $1 \le q \le \infty$ defined as in (5.5) and denote by $\dot{L}^q_U(\mathbb{R}^N)$ the closed subspace of $L^q_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{L^q_U(\mathbb{R}^N)}$, that is

$$\|\tau_y\phi - \phi\|_{L^q_U(\mathbb{R}^N)} \to 0 \ \text{ as } \ |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations. Note that $L^q(\mathbb{R}^N) \subset \dot{L}^q_U(\mathbb{R}^N)$ for $1 \le q < \infty$ and for $q = \infty$ we get $L^\infty_U(\mathbb{R}^N) = L^\infty(\mathbb{R}^N)$ and $\dot{L}^\infty_U(\mathbb{R}^N) = BUC(\mathbb{R}^N)$.

In order to obtain sharper results we introduce the *uniform Bessel-Sobolev spaces* $H^{k,q}_U(\mathbb{R}^N)$, with $k \in \mathbb{N}$, as the set of functions $\phi \in H^{k,q}_{loc}(\mathbb{R}^N)$ such that

$$\|\phi\|_{H^{k,q}_U(\mathbb{R}^N)} = \sup_{x \in \mathbb{R}^N} \|\phi\|_{H^{k,q}(B(x,1))} < \infty$$

for $k \in \mathbb{N}$. Then denote by $\dot{H}^{k,q}_U(\mathbb{R}^N)$ a subspace of $H^{k,q}_U(\mathbb{R}^N)$ consisting of all elements which are translation continuous with respect to $\|\cdot\|_{H^{k,q}_U(\mathbb{R}^N)}$, that is

$$\|\tau_y\phi - \phi\|_{H^{k,q}_U(\mathbb{R}^N)} \to 0 \ \text{ as } \ |y| \to 0$$

where $\{\tau_y, y \in \mathbb{R}^N\}$ denotes the group of translations.

Consider the complex interpolation functor denoted by $[\ ,\ ]_\theta$, for $\theta \in (0,1)$, see [16] for details. Then for $1 \le q < \infty$, $k \in \mathbb{N} \cup \{0\}$ and $s \in (k, k+1)$ we define $\theta \in (0,1)$ such that $s = \theta(1+k) + (1-\theta)k$, that is $\theta = s - k$. Then one can define the intermediate spaces as

$$H^{s,q}_U(\mathbb{R}^N) = [H^{k+1,q}_U(\mathbb{R}^N), H^{k,q}_U(\mathbb{R}^N)]_\theta,$$

and

$$\dot{H}^{s,q}_U(\mathbb{R}^N) = [\dot{H}^{k+1,q}_U(\mathbb{R}^N), \dot{H}^{k,q}_U(\mathbb{R}^N)]_\theta.$$

Using Proposition 4.2 in [3] it is easy to see that the sharp embeddings of Bessel spaces translate into

$$\dot{H}^{s,q}_U(\mathbb{R}^N) \subset \begin{cases} \dot{L}^r_U(\mathbb{R}^N), \ s - \frac{N}{q} \ge -\frac{N}{r}, \ 1 \le r < \infty & if \ s - \frac{N}{q} < 0 \\ \dot{L}^r_U(\mathbb{R}^N), \ 1 \le r < \infty & if \ s - \frac{N}{q} = 0 \\ C^\eta_b(\mathbb{R}^N) & if \ s - \frac{N}{q} > \eta \ge 0. \end{cases} \tag{6.1}$$

Now, the Laplace operator was considered in the scale of spaces $H^{s,q}_U(\mathbb{R}^N)$ and $\dot{H}^{s,q}_U(\mathbb{R}^N)$ in [3] where it was proved that $-\Delta$ defines an analytic semigroup. However in the "undotted" spaces the semigroup generated by $-\Delta$ is analytic but not strongly continuous and these spaces are less convenient to use because smooth functions are not dense in them; see [3].

Therefore, the scale above is no more than the complex interpolation scale of Section 3 for $-\Delta$ in $L^q_U(\mathbb{R}^N)$ or $\dot{L}^q_U(\mathbb{R}^N)$ respectively. Also, the negative side of the scale is defined by the interpolation/extrapolation procedure described in Section 3.1. Since this is a complex interpolation scale, for $0 < s < k$, $k \in \mathbb{N}$.

$$\dot{H}^{-s,q}_U(\mathbb{R}^N) = [\dot{L}^q_U(\mathbb{R}^N), \dot{H}^{-k,q}_U(\mathbb{R}^N)]_\theta, \qquad \textit{with } \theta = \frac{s}{k}.$$

It was moreover proved in [3, Theorem 5.3, pg. 290], that $-\Delta$ has bounded imaginary powers, and therefore this scale coincides with the fractional power one; see Remark 3.8.

However, since the uniform Sobolev spaces are not reflexive, even for $q = 2$, we do not get the description of the negative part of the scale in terms of the dual spaces, see Section 3.

Therefore, we start with some description of the negative spaces which complements the results in [3].

Proposition 6.1 *We have that*

$$\dot{L}^p_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{-s,q}_U(\mathbb{R}^N) \qquad \textit{if} \quad s - \frac{N}{q'} \geq -\frac{N}{p'}, \qquad s > 0.$$

Proof. We first assume that $0 \leq s \leq 2$.
i) We know from Section 3 that $\dot{H}^{-s,q}_U(\mathbb{R}^N)$ is the completion of $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$ with the norm $\|(-\Delta + I)^{-1} \cdot \|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)}$. This means that $f \in \dot{H}^{-s,q}_U(\mathbb{R}^N)$ if and only if there exists an approximating sequence $\{f_n\} \in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$ that converges to f in $\dot{H}^{-s,q}(\mathbb{R}^N)$.

Since $(-\Delta + I)^{-1}$ is an isometry from $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$ to $\dot{H}^{-s,q}_U(\mathbb{R}^N)$, see the beginning of Section V.1.3 in [1], this is equivalent to

$$(-\Delta + I)^{-1} f_n \longrightarrow (-\Delta + I)^{-1} f \quad \text{in } \dot{H}^{2-s,q}(\mathbb{R}^N),$$

and observe that since $f_n \in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$ then $(-\Delta + I)^{-1} f_n \in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$. Thus, we get that $f \in \dot{H}^{-s,q}_U(\mathbb{R}^N)$ if and only if there exists $\{u_n\} \in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$ such that $u_n \to (-\Delta + I)^{-1} f$ in $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$.
ii) Now, take $f \in \dot{L}^p_U(\mathbb{R}^N)$, then from the results in [3] we have $u = (-\Delta + I)^{-1} f \in \dot{H}^{2,p}_U(\mathbb{R}^N)$ and since $s - \frac{N}{q'} \geq -\frac{N}{p'}$ holds by assumption, we have $\dot{H}^{2,p}_U(\mathbb{R}^N) \hookrightarrow \dot{H}^{2-s,q}_U(\mathbb{R}^N)$, and $2 - s \geq 0$. Therefore $u \in \dot{H}^{2-s,q}_U(\mathbb{R}^N)$.

Since $\dot{H}^{4-s,q}_U(\mathbb{R}^N)$ is dense in $\dot{H}^{2-s,q}_U(\mathbb{R}^N)$, there exist $u_n \in \dot{H}^{4-s,q}_U(\mathbb{R}^N)$ such that $\|u_n - u\|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)} \stackrel{n\to\infty}{\to} 0$ and therefore by i), $f \in \dot{H}^{-s,q}_U(\mathbb{R}^N)$. Note that the inclusion is continuous, since $(-\Delta + I)^{-1}$ is an isometry on the scale and then

$$\|f\|_{\dot{H}^{-s,q}_U(\mathbb{R}^N)} = \|(-\Delta + I)^{-1} f\|_{\dot{H}^{2-s,q}_U(\mathbb{R}^N)} \leq C\|(-\Delta + I)^{-1} f\|_{\dot{H}^{2,p}_U(\mathbb{R}^N)} = C\|f\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

In order to prove the result for $s \geq 0$, we can repeat the whole argument above, using $(-\Delta + I)^{-n}$, which is an isometry on the scale, for a suitable n. If $2 \leq s \leq 4$

we use $n = 2$, thus in part i) we obtain that $f \in \dot{H}_U^{-s,q}(\mathbb{R}^N)$ if there exists a sequence $\{u_n\} \in \dot{H}_U^{6-s,q}(\mathbb{R}^N)$ converging to $u = (-\Delta + I)^{-2} f$ in $\dot{H}_U^{4-s,q}(\mathbb{R}^N)$. In part ii) we now have $u \in \dot{H}_U^{4,p}(\mathbb{R}^N) \hookrightarrow \dot{H}_U^{4-s,q}(\mathbb{R}^N)$ since now $4 - s \geq 0$ and the result follows as before.

In the same way, for $2(k-1) \leq s \leq 2k$, we use $n = k$ and repeat the argument above.

■

Remark 6.2 *Note that the embedding in Proposition 6.1 is precisely the one one could expect from (6.1) if the spaces where reflexive. Also this is the embedding that holds for the standard Bessel scale as in Section 5. Needless to say the conditions for the embeddings read also* $s \geq \frac{N}{p} - \frac{N}{q}$.

Using the spaces above and the convolution with the heat kernel, it was proved in Proposition 2.1, Theorem 2.1 and Theorem 5.3 in [3] that the heat equation defines an order preserving analytic semigroup in $L_U^q(\mathbb{R}^N)$ and , for $1 \leq q < \infty$, which is strongly continuous in $\dot{L}_U^q(\mathbb{R}^N)$ and in $E_\alpha := \dot{H}_U^{2\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$. Moreover, this semigroup satisfies the smoothing estimates

$$\|S_{-\Delta}(t)u_0\|_{\dot{L}_U^r(\mathbb{R}^N)} \leq \frac{M_{r,q}e^{\mu t}}{t^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot{L}_U^q(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}_U^q(\mathbb{R}^N)$$

for $1 \leq q \leq r \leq \infty$ for $\mu > 0$ arbitrary, and

$$\|S_{-\Delta}(t)u_0\|_{\dot{H}_U^{2\alpha,q}(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}} \|u_0\|_{\dot{H}_U^{2\beta,q}(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{H}_U^{2\beta,q}(\mathbb{R}^N)$$

with $\mu > 0$ arbitrary, for any $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \beta$.

It was also proved in [3] using a parabolic argument that type$(-\Delta) = 0$ in the $\dot{L}_U^q(\mathbb{R}^N)$ spaces (and thus in $\dot{H}_U^{\alpha,q}(\mathbb{R}^N)$), which explains why $\mu > 0$ above is arbitrary.

We now show some relevant information on the spectrum and resolvent of $-\Delta$ and Δ^2 in the uniform spaces which is analogous to Lemma 5.1.

Proposition 6.3 *i) For* $1 < q < \infty$, *in the space* $E_0 := \dot{L}_U^q(\mathbb{R}^N)$ *the operator* $-\Delta$ *with domain* $E_1 := D(-\Delta) = \dot{H}_U^{2,q}(\mathbb{R}^N)$, *satisfies the estimate*

$$\|(-\Delta - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leq M|\lambda|^{-1}$$

for all λ *in a sector* $S_{0,\phi}$ *as in (3.1) for* $\phi > 0$ *arbitrarily small.*

Furthermore, $\sigma(-\Delta) = [0, \infty)$, *and thus, type*$(-\Delta) = 0$.

ii) For $1 < q < \infty$, *in the space* $E_0 := \dot{L}_U^q(\mathbb{R}^N)$ *the operator* Δ^2 *with domain* $E_2 := D(\Delta^2) = \dot{H}_U^{4,q}(\mathbb{R}^N)$, *satisfies the estimate*

$$\|(\Delta^2 - \lambda)^{-1}\|_{\mathcal{L}(E_0)} \leq M|\lambda|^{-1}$$

for all λ *in a sector* $S_{0,2\phi}$ *as in (3.1) for* $\phi > 0$ *arbitrarily small.*

Furthermore, $\sigma(\Delta^2) = [0, \infty)$, *and thus, type*$(\Delta^2) = 0$.

Proof. To prove part i), observe that, as in page 32–33 in [11], we can obtain an expression for the operator $(-\Delta + \mu I)^{-1}$, provided $Re(\sqrt{\mu}) > 0$, as a convolution operator. The expression is

$$u = (-\Delta + \mu)^{-1} f = \Gamma_\mu * f, \qquad Re(\sqrt{\mu}) > 0$$

with

$$\Gamma_\mu(x) = \sqrt{\mu}^{N-2} G_2(\sqrt{\mu} x), \qquad x \in \mathbb{C}^N, \qquad Re(\sqrt{\mu}) > 0$$

where G_2 is defined as

$$G_2(x) = \frac{1}{(4\pi)^{N/2}} \int_0^\infty t^{-N/2} e^{-t + x\cdot x/4t} dt, \qquad x \in \mathbb{C}^N,$$

see page 132 in [15] or page 33 in [11].

According to [11], we have for $z \in \mathbb{C}^N$ and $N > 2$ and $Re(\xi) > 0$

$$|G_2(z)| \le C|\xi|^{(2-N)/2} (Re\ \xi)^{(2-N)/2} e^{-\frac{1}{2} Re\xi} \qquad \xi = \sqrt{z \cdot z} \tag{6.2}$$

and if $N = 2$,

$$|G_2(z)| \le C \max\{\ln \frac{1}{Re\xi}, 1\} e^{-\frac{1}{2} Re\xi} \qquad \xi = \sqrt{z \cdot z}. \tag{6.3}$$

Now observe that if $\lambda \in S_{0,\phi}$ with $\phi > 0$ then for $\mu = -\lambda \in \mathbb{C} \setminus (-\infty, 0]$ we can choose $Re(\sqrt{\mu}) > 0$. For such λ and similarly to Lemma 5.1 we are going to check that for $f \in \dot{L}^q_U(\mathbb{R}^N)$ we have the following estimate for $u = \Gamma_\mu * f$,

$$\|u\|_{L^q_U(\mathbb{R}^N)} \le C \frac{1}{|\lambda|} \|f\|_{L^q_U(\mathbb{R}^N)}, \qquad \lambda \in S_{0,\phi} \qquad \phi > 0.$$

Let $\{Q_i\}$, $i \in \mathbb{Z}^N$, be a partition of $\mathbb{R}^N$ in open disjoint cubes centered in $i \in \mathbb{Z}^N$ with edges of length 1, parallel to the axes. Thus $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $\mathbb{R}^N = \cup_i \overline{Q_i}$.

Then we fix $i \in \mathbb{Z}^N$ and decompose $f \in \dot{L}^q_U(\mathbb{R}^N)$ in a *far* and a *near* region as in Proposition 2.1 in [3]. For this we denote by $N(i)$ the set for indices j such that $\overline{Q_i} \cap \overline{Q_j} \neq 0$. That is, the set for which

$$d_{ij} := \inf\{dist(x, y), x \in Q_i, y \in Q_j\} \tag{6.4}$$

satisfies that $d_{ij} = 0$. Thus we can define, for each $i \in \mathbb{Z}^N$ fixed

$$Q_i^{near} = \cup_{j \in N(i)} Q_j \qquad and \qquad Q_i^{far} = \mathbb{R}^N \setminus Q_i^{near}.$$

Hence, we decompose $f := f_i^{near} + f_i^{far} := f\chi_{Q_i^{near}} + f\chi_{Q_i^{far}}$ and $u := u_i^{near} + u_i^{far}$, with

$$u_i^{near} := \Gamma_\mu * f_i^{near} \qquad u_i^{far} := \Gamma_\mu * f_i^{far}.$$

The resolvent estimate will follow from the following estimates of the two terms of the decomposition. For λ as above, we have first,

$$\|u_i^{near}\|_{L^q(Q_i)} \le \frac{C}{|\lambda|} \|f\|_{L^q(Q_i^{near})}, \qquad \lambda \in S_{0,\phi} \tag{6.5}$$

and, second,

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le \frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}, \qquad \lambda \in S_{0,\phi} \tag{6.6}$$

for some C independent if $i \in \mathbb{Z}^N$.

In fact, since the constants for the embedding $L^\infty(Q_i) \hookrightarrow L^q(Q_i)$ and restrictions $L^q_U(\mathbb{R}^N) \hookrightarrow L^q(Q_i^{near})$, $L^q_U(\mathbb{R}^N) \hookrightarrow L^1_U(Q_i^{near})$ depend on N but can be chosen independent of p, q and i, (6.5) and (6.6) imply

$$\|u\|_{L^q(Q_i)} \le \frac{C}{|\lambda|}\|f\|_{L^q_U(\mathbb{R}^N)}, \qquad \lambda \in S_{0,\phi} \tag{6.7}$$

for each $i \in \mathbb{Z}^N$ with C independent of i and $\lambda \in S_{0,\phi}$, which gives the result.

Hence, we first prove (6.5). As a consequence of Lemma 5.1, we get for all $\lambda \in S_{0,\phi}$

$$\|u_i^{near}\|_{L^q(Q_i)} \le \|u_i^{near}\|_{L^q(\mathbb{R}^N)} \le \frac{C}{|\lambda|}\|f_i^{near}\|_{L^q(\mathbb{R}^N)} = \frac{C(N)}{|\lambda|}\|f\|_{L^q(Q_i^{near})}.$$

We show now (6.6) for $N > 2$. Observe that $f_i^{far} = f\chi_{Q_i^{far}} = \sum_{j\in\mathbb{Z}^N\setminus N(i)} f\chi_{Q_j}$. Hence, because of (6.2) with $z = \sqrt{\mu}x$, $Re(\sqrt{\mu}) > 0$, $x \in \mathbb{R}^N$, $\mu = -\lambda$ and $\lambda \in S_{0,\phi}$, we have for all $x \in Q_i$

$$|u_i^{far}(x)| = \sum_{j\notin N(i)} |(\Gamma_\mu * f\chi_{Q_j})(x)|$$

$$\le \sum_{j\notin N(i)} C \sup_{y\in Q_j} |\sqrt{\mu}^{N-2} \cdot (\sqrt{\mu}|x-y|)^{1-N/2} Re(\sqrt{\mu}|x-y|)^{1-N/2} e^{-\frac{1}{2}Re\sqrt{\mu}|x-y|}|\|f\|_{L^1(Q_j)}$$

$$\le C\|f\|_{L^1_U(Q_i^{far})}\sqrt{|\lambda|}^{N/2-1} Re(\sqrt{\mu})^{1-N/2} \sum_{j\notin N(i)} \sup_{y\in Q_i} |x-y|^{2-N} e^{-\frac{1}{2}|x-y|Re\sqrt{\mu}}.$$

Note that for all $x \in Q_i$ and $y \in Q_j$ it holds $|x-y| \ge d_{ij}$, thus

$$|u_i^{far}(x)| \le C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{j\notin N(i)} d_{ij}^{2-N} e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Hence

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{j\notin N(i)} d_{ij}^{2-N} e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Now, using that $\sharp\{j \in \mathbb{Z}, d_{ij} = k\} \le Ck^{N-1}$ we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \sum_{k=1}^\infty k e^{-\frac{1}{2}kRe\sqrt{\mu}}$$

$$\le C\|f\|_{L^1_U(Q_i^{far})}\Big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\Big)^{N/2-1} \int_1^\infty x e^{-\frac{1}{2}xRe\sqrt{\mu}}\,dx.$$

Finally, changing variables in the integral above as $y = Re(\sqrt{\mu})x$, we obtain

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le C\big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^{N/2-1}\frac{1}{Re(\sqrt{\mu})^2}\|f\|_{L^1_U(Q_i^{far})}$$

which can be arranged as

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le \big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^{N/2+1}\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

To conclude, observe that for all $\lambda \in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le \frac{C}{\cos(\phi/2)^{N/2+1}}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus, (6.6) is proved for $N > 2$.

We show now (6.6) for $N = 2$. Proceeding as above and using (6.3) we get

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le C\|f\|_{L^1_U(Q_i^{far})}\sum_{j\notin N(i)} \max\{\ln\frac{1}{d_{ij}Re(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}d_{ij}Re\sqrt{\mu}}.$$

Using again that $\sharp\{j \in \mathbb{Z}, d_{ij} = k\} \le Ck^{N-1}$ we get

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le C\|f\|_{L^1_U(Q_i^{far})}\sum_{k=1}^{\infty} k\max\{\ln\frac{1}{kRe(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}kRe\sqrt{\mu}}$$

$$\le C\|f\|_{L^1_U(Q_i^{far})}\int_0^\infty x\max\{\ln\frac{1}{xRe(\sqrt{\mu})}, 1\}e^{-\frac{1}{2}xRe\sqrt{\mu}}dx$$

and with the change of variables $y = Re(\sqrt{\mu})x$ we obtain,

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le \|f\|_{L^1_U(Q_i^{far})}\frac{C}{Re(\sqrt{\mu})^2} = \big(\frac{\sqrt{|\lambda|}}{Re(\sqrt{\mu})}\big)^2\frac{C}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}.$$

Thus for all $\lambda \in S_{0,\phi}$ we find

$$\|u_i^{far}\|_{L^\infty(Q_i)} \le \frac{C}{\cos(\phi/2)^2}\frac{1}{|\lambda|}\|f\|_{L^1_U(Q_i^{far})}$$

and the result is proved.

In particular, $\sigma(-\Delta) \subset [0, \infty)$. For the opposite inclusion, note that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$ satisfies $u \in \dot{L}^p_U(\mathbb{R}^N)$ and

$$-\Delta u = \lambda u$$

for $\lambda = |\omega|^2 \subset [0, \infty)$, and thus $[0, \infty) \subset \sigma(-\Delta)$.

For part ii), since $-\Delta$ is sectorial with sector $S_{0,\phi}$ with $\phi < \pi/4$ and we have the estimate $\|(-\Delta - \lambda)^{-1}\| \le \frac{C}{|\lambda|}$ for $\lambda \in S_{0,\phi}$, we apply Proposition 4.1. Therefore, we get that Δ^2 is sectorial with sector $S_{0,2\phi}$. Note that $\sigma(\Delta^2) \subset [0,\infty)$ because $\phi > 0$ is arbitrarily small. Also, note again that $u(x) = e^{i\omega x}$, $\omega \in \mathbb{R}^N$ satisfies $u \in \dot{L}^p_U(\mathbb{R}^N)$ and

$$\Delta^2 u = \lambda u$$

for $\lambda = |\omega|^4 \subset [0,\infty)$. ■

Now we are ready to use Proposition 4.3 and an argument as in Lemma 5.2 to get the next result.

Lemma 6.4 *Consider the problem*

$$\begin{cases} u_t + \Delta^2 u = 0 & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases} \tag{6.8}$$

i) Then for each $1 < q < \infty$, (6.8) defines an analytic semigroup, $S_{\Delta^2}(t)$, in the scale $X_\alpha := E_{2\alpha} = \dot{H}^{4\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that

$$\|S_{\Delta^2}(t)u_0\|_{\dot{H}^{4\alpha,q}_U(\mathbb{R}^N)} \leq \frac{M_{\alpha,\beta}e^{\mu t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N)$$

with $\alpha, \beta \in \mathbb{R},\ \alpha \geq \beta$.
ii) The analytic semigroup $S_{\Delta^2}(t)$, in $\dot{L}^q_U(\mathbb{R}^N)$, $1 < q < \infty$, satisfies

$$\|S_{\Delta^2}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu_0 t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for any $1 < q \leq r \leq \infty$ and $\mu_0 > 0$ and some $M_{q,r} > 0$.

For a similar estimate with $q = 1 < r \leq \infty$, see Remark 5.3.

We can now adapt the arguments for Bessel and Lebesgue spaces in Section 5 to the uniform Bessel spaces to perturb equation (6.8) as follows. First, as in [14, Lemma 26, pg. 43] we have

Lemma 6.5 *i) Assume that $m \in L^p_U(\mathbb{R}^N)$, then the multiplication operator*

$$Pu(x) = m(x)u(x)$$

satisfies, for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(L^r_U(\mathbb{R}^N), L^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(L^r_U(\mathbb{R}^N),L^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

ii) If moreover $m \in \dot{L}^p_U(\mathbb{R}^N)$ we have for $r \geq p'$ and $\frac{1}{s} = \frac{1}{r} + \frac{1}{p}$, that

$$P \in \mathcal{L}(\dot{L}^r_U(\mathbb{R}^N), \dot{L}^s_U(\mathbb{R}^N)), \qquad \|P\|_{\mathcal{L}(\dot{L}^r_U(\mathbb{R}^N),\dot{L}^s_U(\mathbb{R}^N))} \leq C\|m\|_{L^p_U(\mathbb{R}^N)}.$$

Then Theorem 2.1 leads to

Lemma 6.6 *Let $m \in \dot{L}^p_U(\mathbb{R}^N)$ be such that $\|m\|_{\dot{L}^p_U(\mathbb{R}^N)} \le R_0$, with $p > \frac{N}{4}$. Then for any $1 < q < \infty$ the problem*

$$\begin{cases} u_t + \Delta^2 u = m(x)u & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & in\ \mathbb{R}^N \end{cases}$$

defines an analytic semigroup $S(t)$ in $\dot{L}^q_U(\mathbb{R}^N)$ that satisfies

$$\|S(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \le \frac{M_{r,q}e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, \quad u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for $1 < q \le r \le \infty$ with $M_{q,r}$ and μ depending on m only through R_0.

Furthermore, if

$$m_\varepsilon \to m \quad in\ \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{4}$$

then for every $1 < q \le r \le \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_\varepsilon(t) - S(t)\|_{\mathcal{L}(\dot{L}^q(\mathbb{R}^N),\dot{L}^r(\mathbb{R}^N))} \le \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \le T.$$

Proof. With $Z_{\alpha(r)} := L^r(\mathbb{R}^N)$, $\alpha(r) = -\frac{N}{4r} \in I := [-\frac{N}{4}, 0]$, we know from Lemma 6.4 that

$$\|S_{\Delta^2}(t)\|_{\mathcal{L}(Z_\beta,Z_\alpha)} \le \frac{C}{t^{\alpha-\beta}} \qquad \beta \le \alpha \qquad \alpha,\beta \in I, \quad 0 < t \le 1$$

and we read Lemma 6.5 as $P \in \mathcal{L}(Z_\alpha, Z_\beta)$, for $\alpha = -\frac{N}{4r}$ and $\beta = \alpha - \frac{N}{4p} = -\frac{N}{4s}$ for any $0 \ge \alpha \ge \alpha_0 = -\frac{N}{4}$, with $0 \le \alpha - \beta = -\frac{N}{4r} + \frac{N}{4s} = \frac{N}{4p} < 1$ since $p > \frac{N}{4}$ and $\|P\|_{\mathcal{L}(Z_\alpha,Z_\beta)} \le C\|m\|_{\dot{L}^p_U(\mathbb{R}^N)}$.

Then we apply Theorems 2.1 and 2.2 for each α, β as above. Note that, arguing as in Lemma 5.5, γ and γ' can be taken in the whole interval $I = [-\frac{N}{4}, 0]$.

Finally, the analyticity will follow from Theorem 6.8 below with $a = 0$. ■

Now, we consider more general perturbations, similar to the perturbations in (5.4) with $b = 0$, that is,

$$P_a u = d(x)D^a u \tag{6.9}$$

with $d \in \dot{L}^p_U(\mathbb{R}^N)$ and $a \in \mathbb{N}$. Note that since the uniform Bessel spaces are not reflexive (even for $q = 2$), the negative spaces cannot be described as dual spaces, and thus, the approach in Proposition 5.9 can not be carried out for $b \ne 0$ in uniform spaces.

Proposition 6.7 *Let $P_a u = d(x)D^a u$ with $d \in \dot{L}^p_U(\mathbb{R}^N)$, $a \in \{0, 1, 2, 3\}$ and let $s \ge a$, $\sigma \ge 0$. Then for $1 < q < \infty$, if*

$$(s - a - \frac{N}{q})_- + (\sigma - \frac{N}{q'})_- \ge -\frac{N}{p'} \tag{6.10}$$

we have

$$P_a \in \mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N), \dot{H}^{-\sigma,q}_U(\mathbb{R}^N)), \qquad \|P_a\|_{\mathcal{L}(\dot{H}^{s,q}_U(\mathbb{R}^N),\dot{H}^{-\sigma,q}_U(\mathbb{R}^N))} \le C\|d\|_{\dot{L}^p_U(\mathbb{R}^N)}.$$

Proof. First note that $u \in \dot H^{s,q}_U(\mathbb{R}^N)$, thus $D^a u \in \dot H^{s-a,q}_U(\mathbb{R}^N)$. Because of (6.10) we can choose $r, \rho \geq 1$ such that $(s-a-\frac{N}{q})_- \geq -\frac{N}{r}$ and $(\sigma - \frac{N}{q'})_- \geq -\frac{N}{\rho'}$ with $\frac{1}{\rho} = \frac{1}{r} + \frac{1}{p}$ (and so $r \geq p'$).

Therefore we can use the inclusion $\dot H^{s-a,q}_U(\mathbb{R}^N) \hookrightarrow \dot L^r_U(\mathbb{R}^N)$ and then part ii) in Lemma 6.5 gives $P_a u \in \dot L^\rho_U(\mathbb{R}^N)$ and finally, because of Proposition 6.1, we use the inclusion $\dot L^\rho_U(\mathbb{R}^N) \hookrightarrow \dot H^{-\sigma,q}_U(\mathbb{R}^N)$ and we get the result. ∎

With this, we can obtain the main result for perturbations of (6.8).

Theorem 6.8 *Let $d \in \dot L^p_U(\mathbb{R}^N)$ such that $\|d\|_{\dot L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{4-a}$, $a \in \{0,1,2,3\}$, then for any $1<q<\infty$ and any P_a as in (6.9) there exists an interval $I(q,a) \subset (-1+\frac{a}{4}, 1)$ containing $(-1+\frac{a}{4}+\frac{N}{4p}, 1-\frac{N}{4p})$, such that for any $\gamma \in I(q,a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$ in the space $\dot H^{4\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + \Delta^2 u + d(x) D^a u = 0, & x \in \mathbb{R}^N,\ t>0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\|S_{P_a}(t)u_0\|_{\dot H^{4\gamma',q}_U(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma} e^{\mu t}}{t^{\gamma'-\gamma}} \|u_0\|_{\dot H^{4\gamma,q}_U(\mathbb{R}^N)}, \quad t>0,\ u_0 \in \dot H^{4\gamma}_U(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_a}(t)u_0\|_{\dot L^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r} e^{\mu t}}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}} \|u_0\|_{\dot L^q_U(\mathbb{R}^N)}, \quad t>0,\ u_0 \in \dot L^q_U(\mathbb{R}^N)$$

for $1<q\leq r\leq\infty$ with some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

For each P_a, the interval $I(q,a)$ is given by

$$I(q,a) = (-1+\frac{a}{4}+\frac{N}{4}(\frac{1}{p}-\frac{1}{q'})_+, 1-\frac{N}{4}(\frac{1}{p}-\frac{1}{q})_+) \subset (-1+\frac{a}{4}, 1).$$

Finally, if, as $\varepsilon \to 0$

$$d_\varepsilon \to d \quad \text{in } \dot L^p_U(\mathbb{R}^N), \quad p > \frac{N}{4-k}$$

then for every $T>0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot H^{4\gamma,q}_U(\mathbb{R}^N), \dot H^{4\gamma',q}_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0<t\leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma \geq \gamma'$ and for all $1<q\leq r\leq\infty$,

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot L^q_U(\mathbb{R}^N), \dot L^r_U(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{4}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0<t\leq T.$$

Proof. The proof is as in proof of Proposition 5.10 but using Proposition 6.7 instead of Proposition 5.9. The analyticity comes again from part i) in Theorem 2.3. ∎

Remark 6.9 *We can replace D^a in (6.9) by $(-\Delta + cI)^{a/2}$ with $0 \leq a < 4$ as in Theorem 5.12.*

7 Some other higher order equations

In this section we show that all the results in Sections 4, 5 and 6 above also hold true for other natural powers of suitable operators, and in particular, for any power of the Laplacian, $(-\Delta)^m$, with $m \in \mathbb{N}$. The proofs below have barely no changes with respect to the ones above, and we now detail the main points for them. We start reviewing the abstract results in Section 4.

Proposition 7.1 *Proposition 4.1 remains true for A_0^m, $m \in \mathbb{N}$, as long as the sector $S_{0,\phi}$ for A_0 has an opening angle $\phi < \frac{\pi}{2m}$.*

In fact, this is the original result in Theorem 10.5 in [12].

Now, for the interpolation scale, similarly to Propositions 4.3, we get

Proposition 7.2 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and assume $A_0^m := A_0 \circ \cdots \circ A_m \in \mathcal{H}(E_m, E_0)$, $m \in \mathbb{N}$. Then on the interpolation scale $X_\alpha = E_{m\alpha}$ with $\alpha \in \mathbb{R}$ we have $A_\alpha^m := A_\alpha \circ \cdots \circ A_{\alpha+m} \in \mathcal{H}(X_{\alpha+m}, X_\alpha)$ and A_0^m defines a semigroup $S_{A_0^m}(t)$ in $\{X_\alpha\}_{\alpha\in\mathbb{R}}$ such that $S_{A_0^m}(t)|_{X_\alpha} = e^{-A_\alpha^m t}$ and*

$$\|S_{A_0^m}(t)\|_{\mathcal{L}(X_\beta, X_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\beta-\alpha}} e^{\mu t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta$$

for any $\mu > type(A_0^m)$. The constant $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$X_{-\alpha} = (X_\alpha^\sharp)' \quad \text{and} \quad A_{-\alpha}^m = (A_\alpha^{m\sharp})', \qquad \alpha > 0$$

and it holds that

$$e^{-A_{-\alpha}^m t} = (e^{-A_\alpha^{m\sharp}})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A_\alpha^m u = 0, & t > 0 \\ u(0) = u_0 \in X_\alpha \end{cases}$$

for $\alpha \in \mathbb{R}$ has a unique solution $u(t) = S_{A_0^m}(t) = e^{-A_\alpha^m t} u_0$.

On the other hand, for the fractional power scale, as in Proposition 4.4, we get

Proposition 7.3 *Let $A_0 \in \mathcal{H}(E_1, E_0)$ and assume $A_0^m := A_0 \circ \cdots \circ A_m \in \mathcal{H}(E_m, E_0)$. Also, fix $N \in \mathbb{N}$. Then on the fractional power scale $Y_\alpha = F_{m\alpha}$ with $\alpha \geq -N$ we have $A_\alpha^m := A_\alpha \circ \cdots \circ A_{\alpha+m} \in \mathcal{H}(Y_{\alpha+m}, Y_\alpha)$ and A_0^m defines a semigroup $S_{A_0^m}(t)$ in $\{Y_\alpha\}_{\alpha\geq -N}$ such that $S_{A_0^m}(t)|_{Y_\alpha} = e^{-A_\alpha^m t}$ and*

$$\|S_{A_0^m}(t)\|_{\mathcal{L}(Y_\beta, Y_\alpha)} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu t} \qquad t > 0,\ \alpha \geq \beta \geq -N$$

for any $\mu > type(A_0^m)$. The constant $C(\alpha - \beta)$ is bounded for α, β in bounded sets of $\mathbb{R}$.

If E_0 is reflexive, the negative side of the scale can be described as

$$Y_{-\alpha} = (Y_\alpha^\sharp)' \quad \text{and} \quad A^m_{-\alpha} = (A_\alpha^{m\sharp})' \qquad \alpha > 0.$$

and it holds that

$$e^{-A^m_{-\alpha} t} = (e^{-A_\alpha^{m\sharp}})'.$$

Furthermore, the problem

$$\begin{cases} u_t + A^m_\alpha u = 0, & t > 0 \\ u(0) = u_0 \in Y_\alpha \end{cases}$$

for $\alpha \geq -N$ has a unique solution $u(t) = S_{A_0^m}(t) = e^{-A^m_\alpha t} u_0$.

The proofs for both propositions follow the same steps as for Propositions 4.3 and 4.4, but replacing Δ^2 by $(-\Delta)^m$. Note that when shifting the operator $\tilde{A}_0 = A_0 + cI$, the perturbation P obtained in the proof of Propositions 4.3 and 4.4 is different (given by the binomial theorem), but the same argument can be repeated.

We now consider powers of the Laplacian in the standard $L^q(\mathbb{R}^N)$ spaces. The following result is similar to Lemma 5.1. Note that $(-\Delta)^m$ has bounded imaginary powers (see Remark 4.5), thus the fractional and interpolation scales coincide.

Lemma 7.4 *For $1 < q < \infty$, in $E_0 = L^q(\mathbb{R}^N)$ the operator $(-\Delta)^m$ with domain $E_m = D(-\Delta^m) = H^{2m,q}(\mathbb{R}^N)$, satisfies the estimate*

$$\|((-\Delta)^m - \lambda)^{-1}\|_{L^q(\mathbb{R}^N)} \leq M|\lambda|^{-1} \qquad \text{for all } \lambda \in S_{0,m\phi}$$

where $\phi > 0$ is arbitrarily small. Furthermore $\sigma((-\Delta)^m) = [0, \infty)$ and therefore

$$\mathit{type}((-\Delta)^m) = 0.$$

The proof is exactly as the one in Lemma 5.1, but using Proposition 7.1 instead of Proposition 4.1. This information, together with Proposition 7.2 leads to

Lemma 7.5 *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0 & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{7.1}$$

with $m \in \mathbb{N}$.
i) Then for $1 < q < \infty$, (7.1) defines an analytic semigroup, $S_{(-\Delta)^m}(t)$, in the scale $X_\alpha = E_{m\alpha} = H^{2m\alpha,q}(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists $C(\alpha - \beta)$ such that

$$\|S_{(-\Delta)^m}(t)\|_{\mathcal{L}(H^{2m\beta,q}(\mathbb{R}^N), H^{2m\alpha,q}(\mathbb{R}^N))} \leq \frac{C(\alpha-\beta)}{t^{\alpha-\beta}} e^{\mu_0 t} \qquad t > 0,\ \alpha, \beta \in \mathbb{R},\ \alpha \geq \beta.$$

ii) The analytic semigroup, $S_{(-\Delta)^m}(t)$, in $L^q(\mathbb{R}^N)$, $1 < q < \infty$, satisfies that for any $\mu_0 > 0$ there exists $M_{q,r}$ such that

$$\|S_{(-\Delta)^m}(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{M_{q,r}}{t^{\frac{N}{2m}(\frac{1}{q} - \frac{1}{r})}} e^{\mu_0 t} \qquad t > 0$$

for $1 < q \leq r \leq \infty$.

Note that the proof in Lemma 5.2 can be carried out now taking $(-\Delta)^m$ instead of Δ^2 in the scale of spaces.

Also note that the solution of problem (7.1) can also be described as the convolution of the initial data with the fundamental kernel for the m–Laplacian operator, which satisfies suitable Gaussian bounds; see e.g. [8, 6].

We can now add the perturbations to (7.1), as in Theorem 5.10.

Theorem 7.6 *Let a, $b \in \mathbb{N}$ with $k = a + b \leq 2m - 1$ and $P_{a,b}$ be as in (5.4). Assume that $\|d\|_{L^p_U(\mathbb{R}^N)} \leq R_0$ with $p > \frac{N}{2m-k}$. Then for any $1 < q < \infty$ and such $P_{a,b}$ there exists an interval $I(q, a, b) \subset (-1 + \frac{a}{2m}, 1 - \frac{b}{2m})$ containing $(-1 + \frac{a}{2m} + \frac{N}{2mp}, 1 - \frac{b}{2m} - \frac{N}{2mp})$, such that for any $\gamma \in I(q, a, b)$, we have a strongly continuous, analytic semigroup, $S_{P_{a,b}}(t)$ in the space $H^{2m\gamma,q}(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + D^b(d(x)D^a u) = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimates

$$\|S_{P_{a,b}}(t)u_0\|_{H^{2m\gamma',q}(\mathbb{R}^N)} \leq \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{H^{2m\gamma,q}(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in H^{2m\gamma,q}(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q, a, b)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_{a,b}}(t)u_0\|_{L^r(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{L^q(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in L^q(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

Furthermore, the interval $I(q, a, b)$ is given by

$$I(q, a, b) = (-1 + \frac{a}{2m} + \frac{N}{2m}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{b}{2m} - \frac{N}{2m}(\frac{1}{p} - \frac{1}{q})_+).$$

Finally, if

$$d_\varepsilon \to d \quad \text{in } L^p_U(\mathbb{R}^N), \quad p > \frac{N}{2m-k}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(H^{2m\gamma,q}(\mathbb{R}^N), H^{2m\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q, a, b)$, with $\gamma' \geq \gamma$ and

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(L^q(\mathbb{R}^N), L^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T$$

for all $1 < q \leq r \leq \infty$.

Note that now, the amount of possible combinations of perturbations becomes enormous, however, they can be combined just as explained in Remark 5.11.

We finally turn into the uniform spaces $\dot{L}^q_U(\mathbb{R}^N)$. First of all, we check the information about the spectrum and resolvent set for $(-\Delta)^m$ in $\dot{L}^q_U(\mathbb{R}^N)$, with the same ideas as in Proposition 6.3, that is, using Proposition 7.1 and Remark 4.2.

Lemma 7.7 *For $1 < q < \infty$, the operator $(-\Delta)^m$ in the space $E_0 = \dot{L}^q_U(\mathbb{R}^N)$ with domain $E_m = D((-\Delta)^m) = \dot{H}^{2m,q}_U(\mathbb{R}^N)$, satisfies the estimate*

$$\|((-\Delta)^m - \lambda)^{-1}\|_{\dot{L}^q_U(\mathbb{R}^N)} \le M|\lambda|^{-1}$$

for all λ in a sector $S_{0,m\phi}$ as in (3.1) for $\phi > 0$ arbitrarily small.

Furthermore, $\sigma((-\Delta)^m) = [0,\infty)$, and thus, type$((-\Delta)^m) = 0$.

Again, this information, together with Proposition 7.2 leads to

Lemma 7.8 *Consider the problem*

$$\begin{cases} u_t + (-\Delta)^m u = 0 & x \in \mathbb{R}^N, t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

i)Then for each $1 < q < \infty$, (7.8) defines an analytic semigroup, $S_{(-\Delta)^m}(t)$, in the scale $X_\alpha := E_{m\alpha} = \dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)$, $\alpha \in \mathbb{R}$, such that for any $\mu_0 > 0$ there exists C such that

$$\|S_{(-\Delta)^m}(t)u_0\|_{\dot{H}^{2m\alpha,q}_U(\mathbb{R}^N)} \le \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{\dot{H}^{4\beta,q}_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{H}^{4\beta,q}_U(\mathbb{R}^N)$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \ge \beta$.
ii) The analytic semigroup $S_{(-\Delta)^m}(t)$, in $\dot{L}^q_U(\mathbb{R}^N)$, $1 < q < \infty$, satisfies

$$\|S_{(-\Delta)^m}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \le \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \qquad t > 0, u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

for any $1 < q \le r \le \infty$ and μ_0 and some $M_{q,r} > 0$.

Then adding perturbations as above, we have

Theorem 7.9 *Let $a \in \mathbb{N}$, $a \le 2m-1$ and $\|d\|_{\dot{L}^p_U(\mathbb{R}^N)} \le R_0$ with $p > \frac{N}{2m-a}$, then for any $1 < q < \infty$ and any P_a as in (6.9) there exists an interval $I(q,a) \subset (-1+\frac{a}{2m}, 1)$ containing $(-1+\frac{a}{2m}+\frac{N}{2mp}, 1-\frac{N}{2mp})$, such that for any $\gamma \in I(q,a)$, we have a continuous, analytic semigroup, $S_{P_a}(t)$ in the space $\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)$, for the problem*

$$\begin{cases} u_t + (-\Delta)^m u + d(x)D^a u = 0, & x \in \mathbb{R}^N,\ t > 0 \\ u(0) = u_0 & \text{in } \mathbb{R}^N. \end{cases}$$

Moreover the semigroup has the smoothing estimate

$$\|S_{P_a}(t)u_0\|_{\dot{H}^{2m\gamma',q}_U(\mathbb{R}^N)} \le \frac{M_{\gamma',\gamma}e^{\mu t}}{t^{\gamma'-\gamma}}\|u_0\|_{\dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)}, \quad t > 0,\ u_0 \in \dot{H}^{2m\gamma,q}_U(\mathbb{R}^N)$$

for every $\gamma, \gamma' \in I(q,a)$ with $\gamma' \geq \gamma$, and

$$\|S_{P_a}(t)u_0\|_{\dot{L}^r_U(\mathbb{R}^N)} \leq \frac{M_{q,r}e^{\mu t}}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}\|u_0\|_{\dot{L}^q_U(\mathbb{R}^N)}, \quad t>0, \quad u_0 \in \dot{L}^q_U(\mathbb{R}^N)$$

with $1 < q \leq r \leq \infty$ and some $M_{\gamma',\gamma}$, $M_{q,r}$ and $\mu \in \mathbb{R}$ depending on d only through R_0.

For each P_a, the interval $I(q,a)$ is given by

$$I(q,a) = (-1 + \frac{a}{2m} + \frac{N}{2m}(\frac{1}{p} - \frac{1}{q'})_+, 1 - \frac{N}{2m}(\frac{1}{p} - \frac{1}{q})_+) \subset (-1 + \frac{a}{2m}, 1).$$

Finally, if

$$d_\varepsilon \to d \quad in\ \dot{L}^p_U(\mathbb{R}^N), \quad p > \frac{N}{2m-k}$$

then for every $1 < q < \infty$ and $T > 0$ there exists $C(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{H}^{2m\gamma,q}(\mathbb{R}^N),\dot{H}^{2m\gamma',q}(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\gamma'-\gamma}}, \qquad \forall\, 0 < t \leq T$$

for all $\gamma, \gamma' \in I(q,a,b)$, $\gamma \geq \gamma'$ and

$$\|S_{P_\varepsilon}(t) - S_P(t)\|_{\mathcal{L}(\dot{L}^q(\mathbb{R}^N),\dot{L}^r(\mathbb{R}^N))} \leq \frac{C(\varepsilon)}{t^{\frac{N}{2m}(\frac{1}{q}-\frac{1}{r})}}, \qquad \forall\, 0 < t \leq T$$

for all $1 < q \leq r \leq \infty$.

The proofs of both Lemma 7.8 and Theorem 7.9 follow the proofs of Lemma 6.4 and Theorem 6.8, just replacing Δ^2 by $(-\Delta)^m$ as the order of the operator involved.

References

[1] H. Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1995. Abstract linear theory.

[2] H. Amann, M. Hieber, and G. Simonett. Bounded H_∞-calculus for elliptic operators. *Differential Integral Equations*, 7(3-4):613–653, 1994.

[3] J.M. Arrieta, J. W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Linear parabolic equations in locally uniform spaces. *Math. Models Methods Appl. Sci.*, 14(2):253–293, 2004.

[4] J.M. Arrieta, J.W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Asymptotic behavior and attractors for reaction diffusion equations in unbounded domains. *Nonlinear Anal.*, 56(4):515–554, 2004.

[5] J.M. Arrieta, J.W. Cholewa, T. Dlotko, and A. Rodríguez-Bernal. Dissipative parabolic equations in locally uniform spaces. *Mathematische Nachrichten*, 280:16435–1663, 2007. Departamento de Matemática Aplicada U. Complutense, Preprint Series MA-UCM 2005-7.

[6] G. Barbatis. Explicit estimates on the fundamental solution of higher-order parabolic equations with measurable coefficients. *J. Differential Equations*, 174(2):442–463, 2001.

[7] J. Cholewa and A. Rodríguez-Bernal. Linear and semilinear higher order parabolic equations in $\mathbb{R}^n$. *Nonlinear Analysis*, 75:194–210, 2012.

[8] N. Dungey. Sharp constants in higher-order heat kernel bounds. *Bull. Austral. Math. Soc.*, 61(2):189–200, 2000.

[9] Yu. V. Egorov, V. A. Galaktionov, V. A. Kondratiev, and S. I. Pohozaev. Global solutions of higher-order semilinear parabolic equations in the supercritical range. *Adv. Differential Equations*, 9(9-10):1009–1038, 2004.

[10] A. Ferrero, F. Gazzola, and H.C. Grunau. Decay and eventual local positivity for biharmonic parabolic equations. *Discrete Contin. Dyn. Syst.*, 21:1129–1157, 2008.

[11] D. Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[12] H. Komatsu. Fractional powers of operators. *Pacific J. Math.*, 19:285–346, 1966.

[13] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag New York, Inc., 1983.

[14] A. Rodríguez-Bernal. Perturbation of analytic semigroups in scales of banach spaces and applications to parabolic equations with low regularity data. *Sema Journal*, 53:3–54, January 2011.

[15] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[16] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2011

1. APPROXIMATING TRAVELLING WAVES BY EQUILIBRIA OF NON LOCAL EQUATIONS, J. M. Arrieta, M. López-Fernández and E. Zuazua.

2. INFINITELY MANY STABILITY SWITCHES IN A PROBLEM WITH SUBLINEAR OSCILLATORY BOUNDARY CONDITIONS, A. Castro and R. Pardo

3. THIN DOMAINS WITH EXTREMELY HIGH OSCILLATORY BOUNDARIES, J́. M. Arrieta and M. C. Pereira

4. FROM NEWTON EQUATION TO FRACTIONAL DIFFUSION AND WAVE EQUATIONS, L. Váquez

5. EL CÁLCULO FRACCIONARIO COMO INSTRUMENTO DE MODELIZACIÓN, L. Váquez and M. P. Velasco

6. THE TANGENTIAL VARIATION OF A LOCALIZED FLUX-TYPE EIGENVALUE PROBLEM, R. Pardo, A. L. Pereira and J. C. Sabina de Lis

7. IDENTIFICATION OF A HEAT TRANSFER COEFFICIENT DEPENDING ON PRESSURE AND TEMPERATURE, A. Fraguela, J. A. Infante, Á. M. Ramos and J. M. Rey

8. A NOTE ON THE LIOUVILLE METHOD APPLIED TO ELLIPTIC EVENTUALLY DEGENERATE FULLY NONLINEAR EQUATIONS GOVERNED BY THE PUCCI OPERATORS AND THE KELLER–OSSERMAN CONDITION, G. Díaz

9. RESONANT SOLUTIONS AND TURNING POINTS IN AN ELLIPTIC PROBLEM WITH OSCILLATORY BOUNDARY CONDITIONS, A. Castro and R. Pardo

10. BE-FAST: A SPATIAL MODEL FOR STUDYING CLASSICAL SWINE FEVER VIRUS SPREAD BETWEEN AND WITHIN FARMS. DESCRIPTION AND VALIDATION. B. Ivorra B. Martínez-López, A. M. Ramos and J.M. Sánchez-Vizcaíno.

11. FRACTIONAL HEAT EQUATION AND THE SECOND LAW OF THERMODYNAMICS, L. Vázquez , J. J. Trujillo and M. P. Velasco

12. LINEAR AND SEMILINEAR HIGHER ORDER PARABOLIC EQUATIONS IN R^N, J. Cholewa and A. Rodríguez Bernal

13. DISSIPATIVE MECHANISM OF A SEMILINEAR HIGHER ORDER PARABOLIC EQUATION IN R^N, J. Cholewa and A. Rodríguez Bernal

14. DYNAMIC BOUNDARY CONDITIONS AS A SINGULAR LIMIT OF PARABOLIC PROBLEMS WITH TERMS CONCENTRATING AT THE BOUNDARY, A. Jimémez-Casas and A. Rodríguez Bernal

15. DISEÑO DE UN MODELO ECONÓMICO Y DE PLANES DE CONTROL PARA UNA EPIDEMIA DE PESTE PORCINA CLÁSICA, E. Fernández Carrión, B. Ivorra, A. M. Ramos, B. Martínez-López, Sánchez-Vizcaíno.

16. BIOREACTOR SHAPE OPTIMIZATION. MODELING, SIMULATION, AND SHAPE OPTIMIZATION OF A SIMPLE BIOREACTOR FOR WATER TREATMENT, J. M. Bello Rivas, B. Ivorra, A. M. Ramos, J. Harmand and A. Rapaport

17. THE PROBABILISTIC BROSAMLER FORMULA FOR SOME NONLINEAR NEUMANN BOUNDARY VALUE PROBLEMS GOVERNED BY ELLIPTIC POSSIBLY DEGENERATE OPERATORS, G. Díaz

PREPUBLICACIONES DEL DEPARTAMENTO
DE MATEMÁTICA APLICADA
UNIVERSIDAD COMPLUTENSE DE MADRID
MA-UCM 2012

1. ON THE CAHN-HILLIARD EQUATION IN H^1(R^N), J. Cholewa and A. Rodríguez Bernal

2. GENERALIZED ENTHALPY MODEL OF A HIGH PRESSURE SHIFT FREEZING PROCESS, N. A. S. Smith, S. S. L. Peppin and A. M. Ramos

3. 2D AND 3D MODELING AND OPTIMIZATION FOR THE DESIGN OF A FAST HYDRODYNAMIC FOCUSING MICROFLUIDIC MIXER, B. Ivorra, J. L. Redondo, J. G. Santiago, P.M. Ortigosa and A. M. Ramos

4. SMOOTHING AND PERTURBATION FOR SOME FOURTH ORDER LINEAR PARABOLIC EQUATIONS IN R^N, C. Quesada and A. Rodríguez-Bernal